Mechanical System/ Modelling of Mechanical Systems- MEC100x – Lectures 3_1

Energy, Power and Intelligent Control

School of Electronics, Electrical Engineering and Computer Science

Ashby Building

Queen's University Belfast

Aims

1. Newton’s laws of motion, Conservation of energy and momentum
2. Kinetic Analysis of simple mechanisms, Forces in Mechanisms, Torque
3. Mathematical models
4. Mechanical system building blocks (Mass/Spring/Damper,…)
5. Modelling dynamic systems
6. First-order systems - Second-order systems

What is System Dynamics?

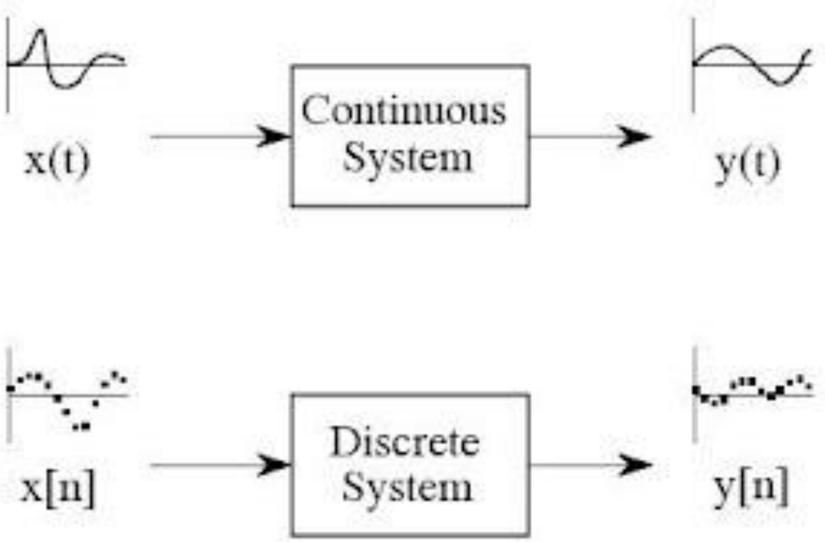
What is System Dynamics?

The synthesis of mathematical models to represent dynamic responses of physical systems for the purpose of analysis, design, and/or control.

System Dynamics draws on a variety of engineering specialties to form a unified approach to study dynamic systems.

Continuous-Time Systems

- Variables and functions defined for all time
- Similar to variables in the "analog" domain
- Described by differential equations



Discrete-Time Systems

- Variables defined only at discrete time points
- Similar to variable in the "digital" domain
- Described by difference equations

http://signalsworld.blogspot.com/2009/11/continuoustime-and-discrete-time.html

Classification of Dynamic Systems

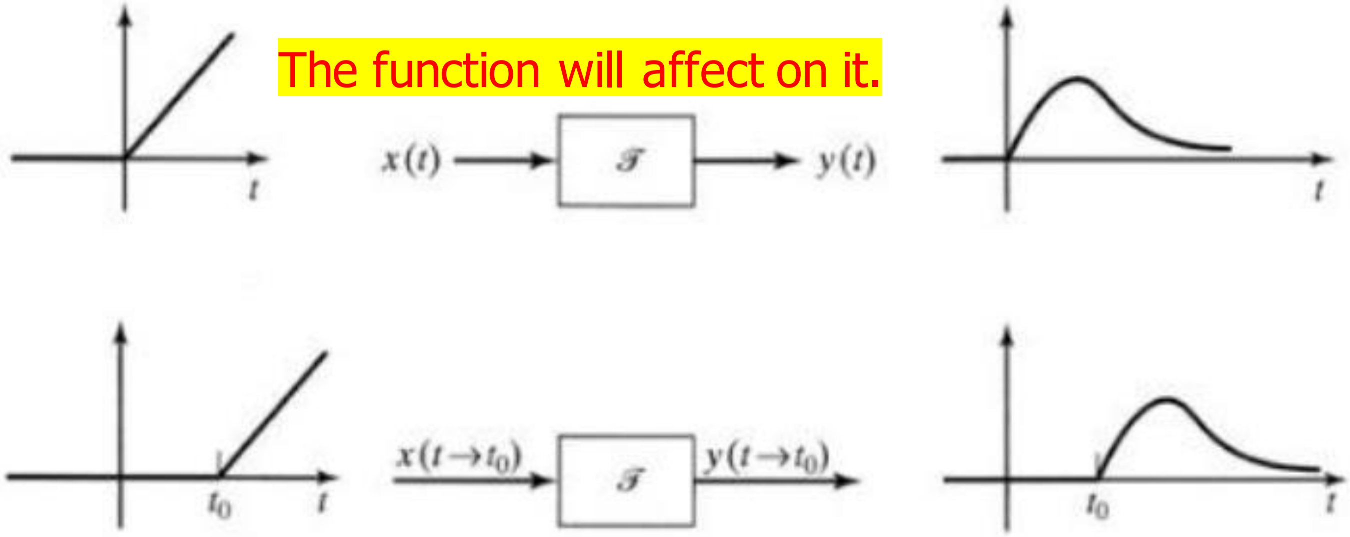
Time-Varying Systems

- System parameters vary with time

Time-Invariant Systems

- System parameters remain constant with time.

The function will affect on it.



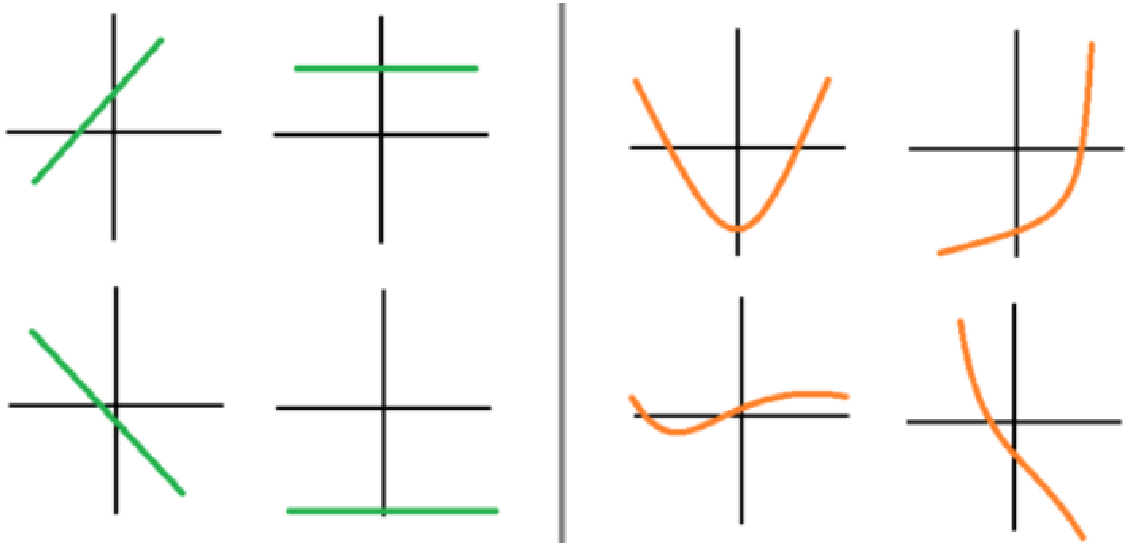
Classification of Dynamic Systems

Linear Systems

- ❑ Obeys superposition
- ❑ Has homogeneity

Non-Linear Systems

- ❑ Does not obey superposition
- ❑ Does not have homogeneity

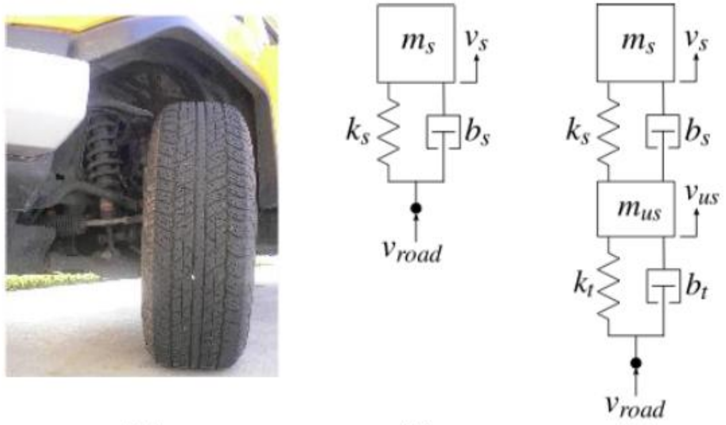


https://study.com/academy/lesson/how-to-recognize-linear-functions-vs-non-linear-functions.html

A Quarter-Car Suspension Model

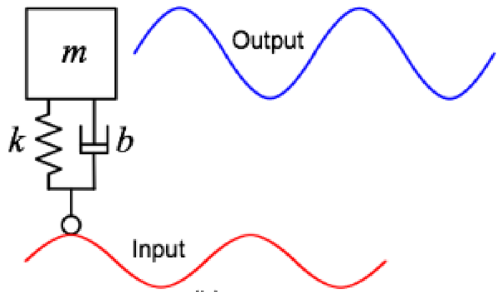
- To formulate a model we must identify the pertinent components and formulate mathematical representations for each.

- The complexity of the model depends on its intended use.

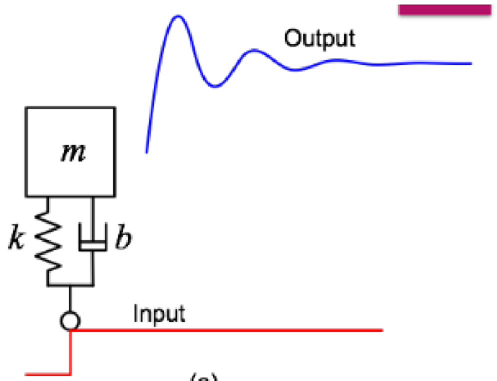


(a) (b) (c)

- We analyze systems to determine what makes them function or respond as they do so that we might be able to alter or optimize their responses.
- Analyses are commonly conducted in the time- or frequency-domains.
- Step responses usually entails time-domain analysis
- Cyclic inputs entails frequency domain analysis



(b)



(a)

- Energy is defined as the capacity for doing work.
- Power is defined as the rate of doing work or the amount of energy consumed per unit time.
- Power (P) is also defined as the multiplication of an effort and a flow

- Effort (e): force-like variable
- Flow, (f): velocity-like variable

Momentum, Effort, Displacement, and Flow

Power $P(t) = e(t)\, f(t)$

Generalized Momentum $p(t) = \int e(t)\, d(t)$

Effort $e(t) = \frac{dp}{dt}$

Generalized Displacement $q(t) = \int f(t)\, d(t)$

Flow $f(t) = \frac{dq}{dt}$

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Potential and Kinetic Energy

Energy

$$E(t) = \int p(t)\, d(t) = \int e(t)\, f(t)\, dt$$

Potential Energy

$$E(t) = \int e(t)\, \frac{dq}{dt} d(t) = \int e(q)\, dq$$

Kinetic Energy

$$E(t) = \int \frac{dp}{dt} f(t)\, d(t) = \int f(p)\, dp$$

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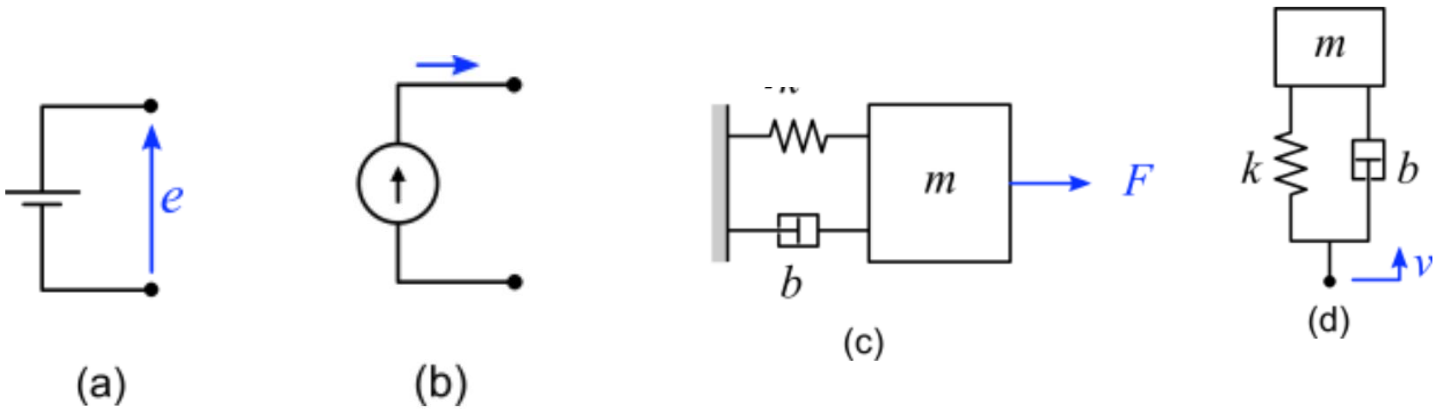
Supply energy

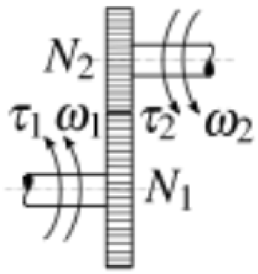
Effort sources specify effort as an input to the system

Flow sources specify flow as an input to the system

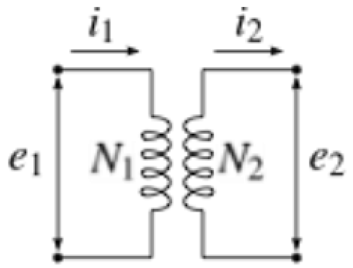
Power goes from the source to the system.



- Converts energy
- Energy-conserving
- Efforts are algebraically related
- Flows are algebraically related
- Power through convention

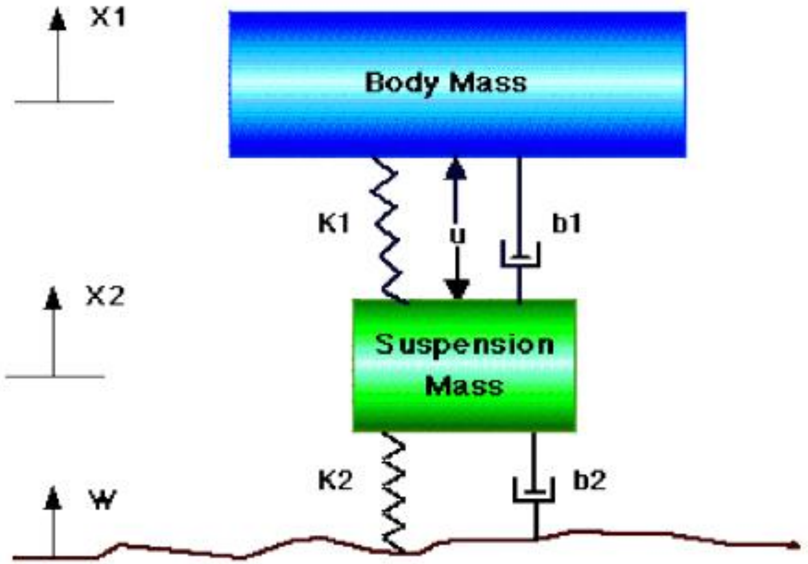


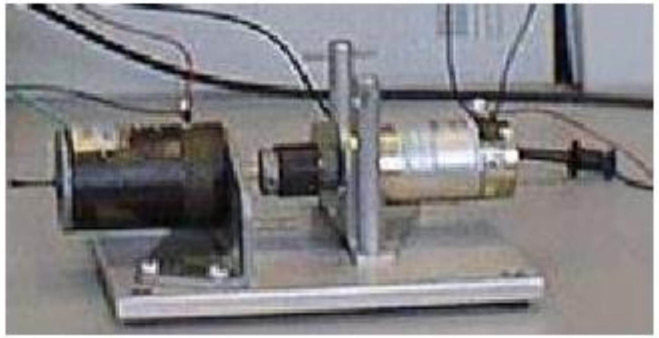
$$n = \frac{\tau_1}{\tau_2} = \frac{\omega_2}{\omega_1}$$



$$\frac{e_1}{e_2} = \frac{i_2}{i_1} = \frac{N_1}{N_2}$$

Model of Bus Suspension System
(1/4 Bus)

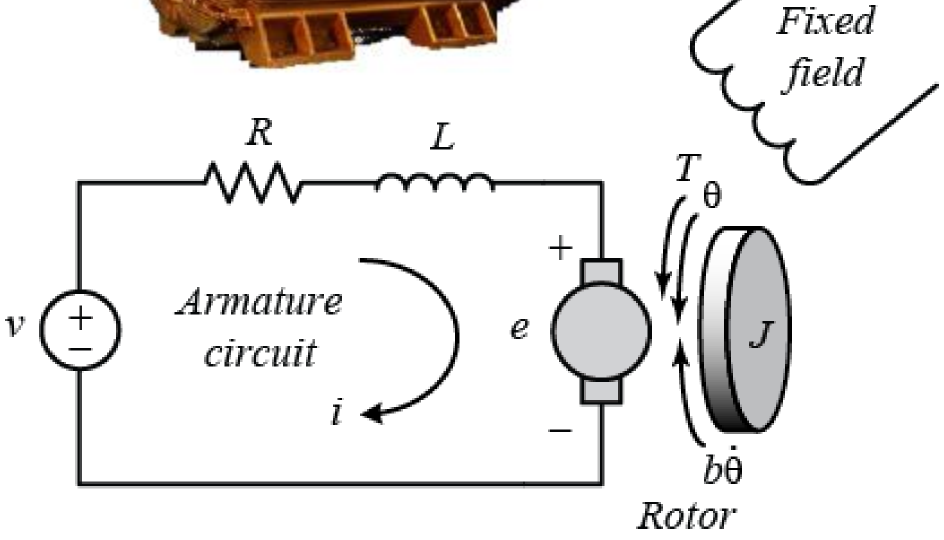




$$\frac{\dot{\theta}}{V} = \frac{K}{(Js+b)(Ls+R)+K^2}$$

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ \frac{1}{L}\end{bmatrix}V$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix}$$



Analysis and Design of Dynamic Systems

- We analyse systems to determine what makes them function or respond as they do so that we might be able to alter or optimize their responses.
- Analyses are commonly conducted in the time- or frequency-domains.
- Step responses usually entails time-domain analysis
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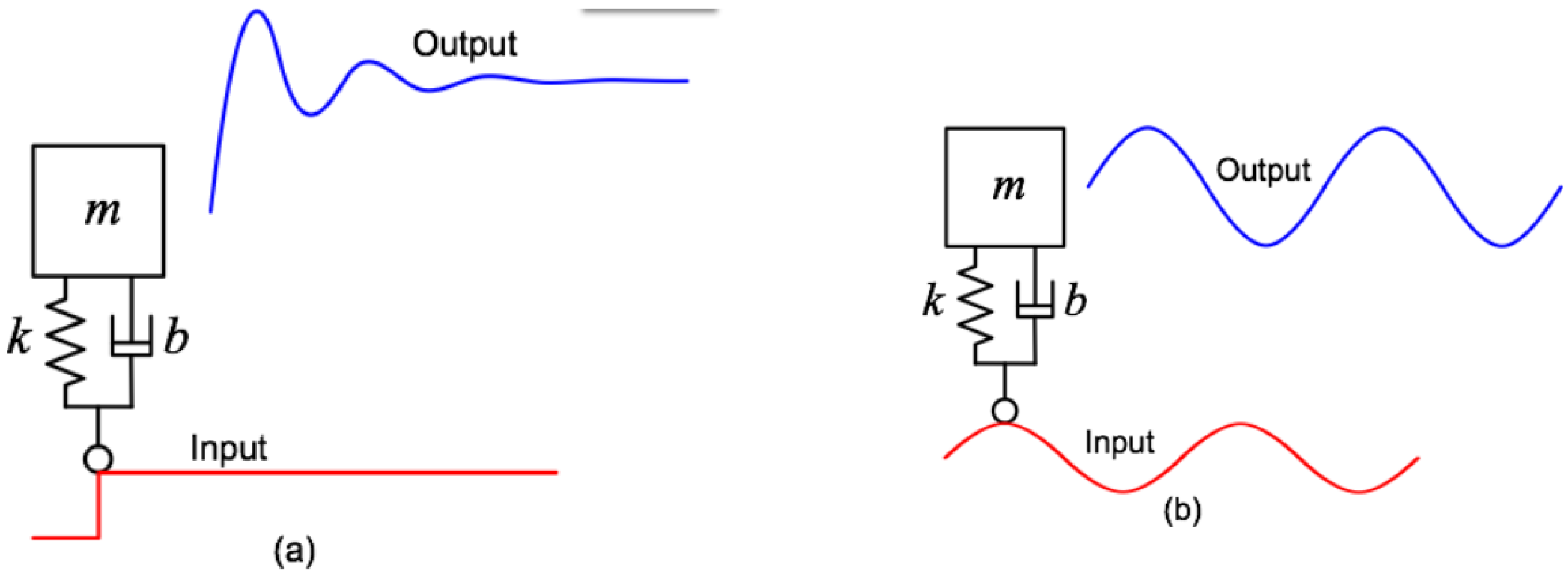


Mechanical system

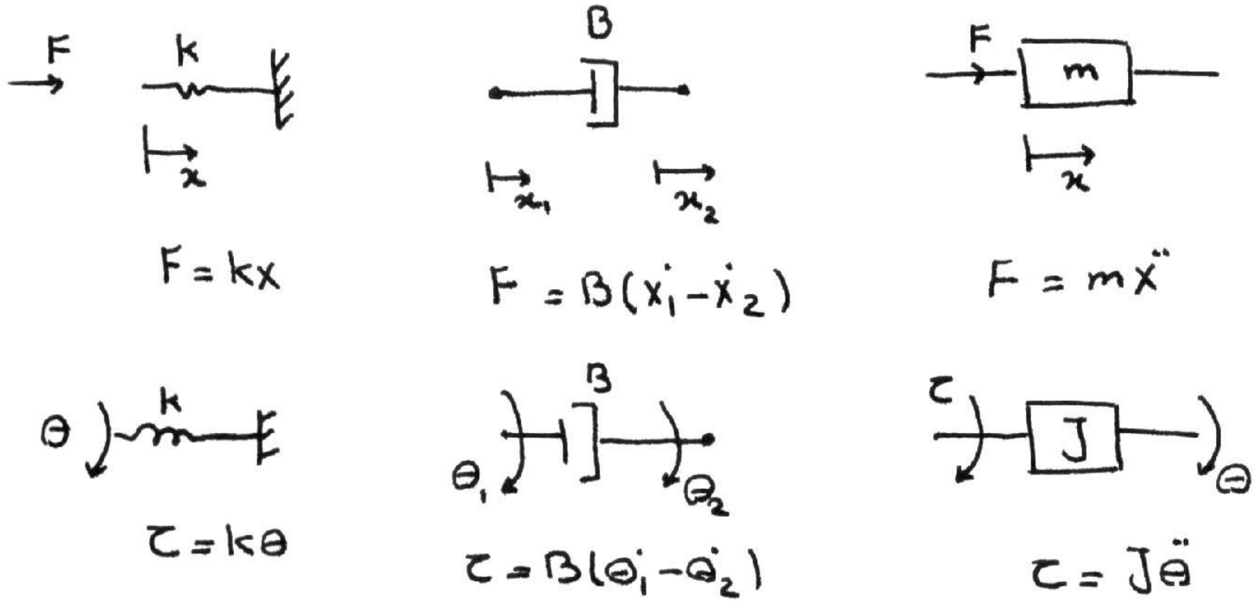
Transitional movement

$$F = ma$$

Rotational movement

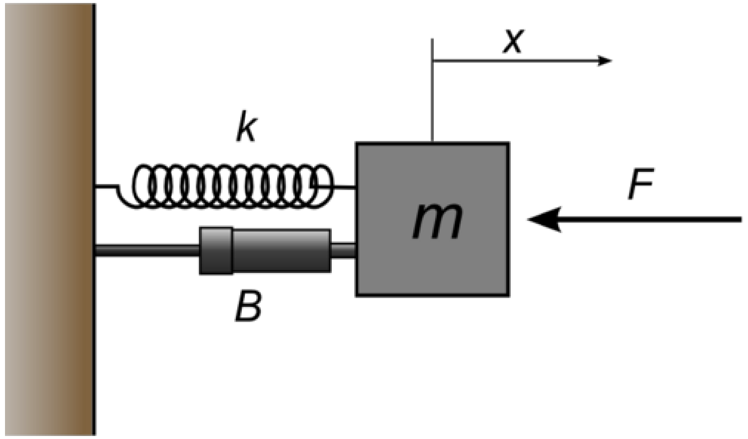
$$\tau = I.\dot{\omega}$$

Mechanical components

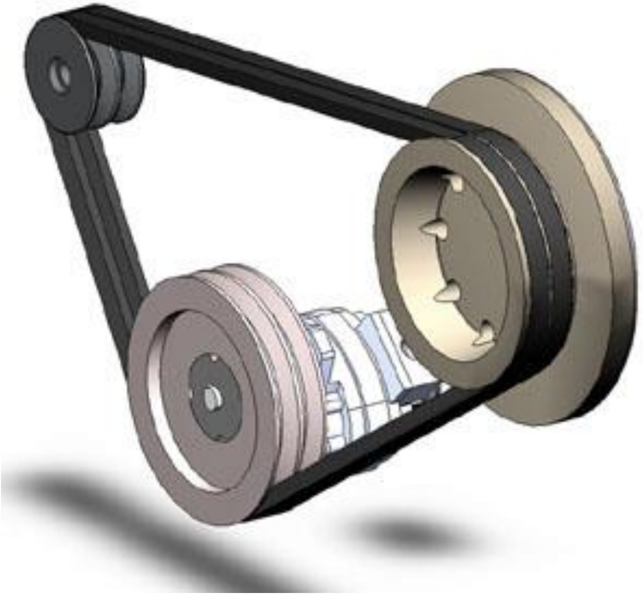


Basic Types of Mechanical Systems

- Translational
  - Linear Motion



- Rotational
  - Rotational Motion



Basic Elements of Translational Mechanical Systems

Translational Spring

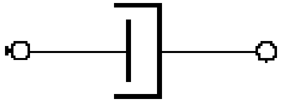
i)

Translational Mass

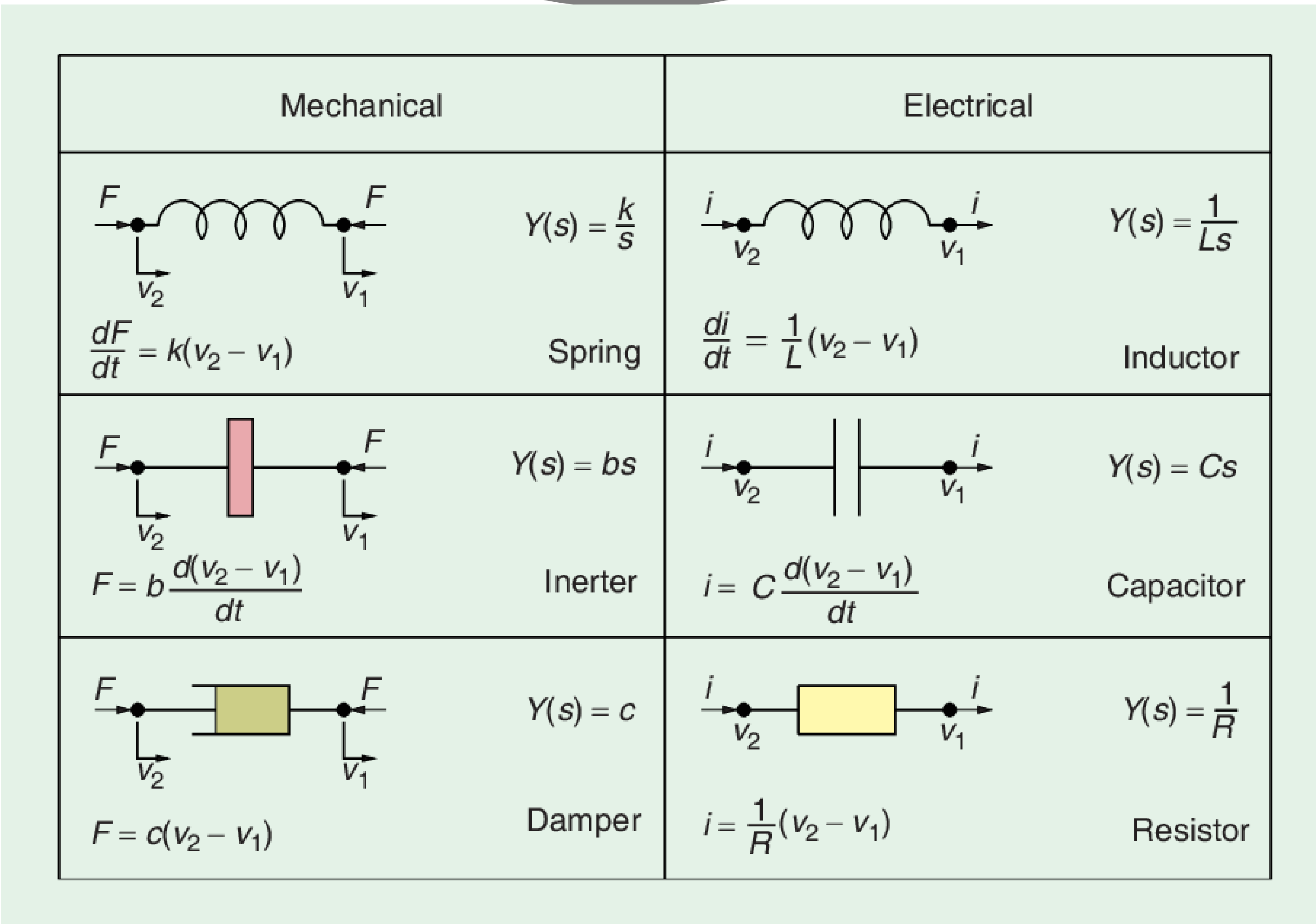
ii)

Translational Damper

iii)



Analogy between the electrical and mechanical elements



https://www.semanticscholar.org/paper/The-missing-mechanical-circuit-element-Chen Papageorgiou/2b61fdd00f6218ab1643750c859b73357c637d2a/figure/0

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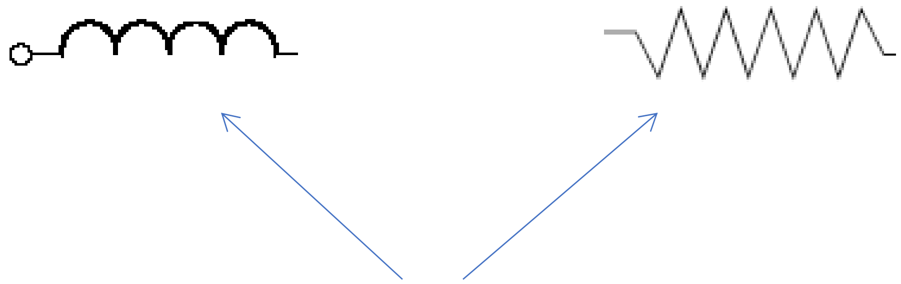
Translational Spring

- A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring

i)

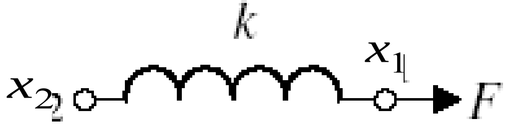


Circuit Symbols

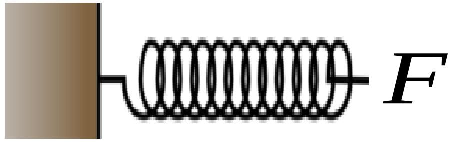
Translational Spring

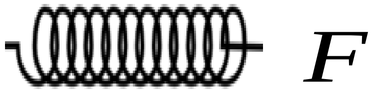
Translational Spring

- If *F* is the applied force



- Then x_1 is the deformation if $x_2 = 0$





- Or $(x_1 - x_2)$ is the deformation.

- The equation of motion is given as

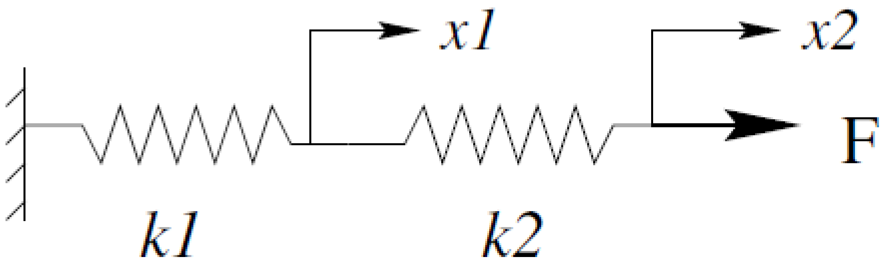
$$F = k\ (x_1 - x_2)$$

- Where k is stiffness of spring expressed in N/m

Translational Spring

- Given two springs with spring constant k_1 and k_2 , obtain the equivalent spring constant k_{eq} for the two springs connected in:

Series



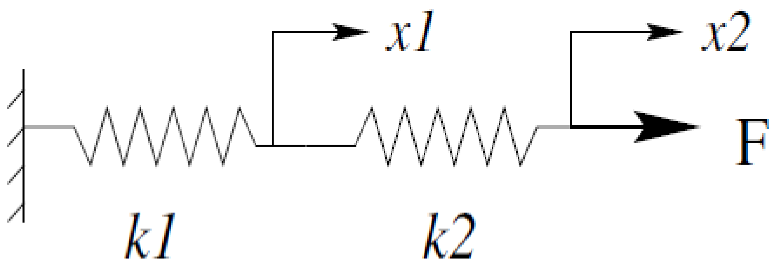
Translational Spring

- The forces on two springs are same, F, however displacements are different therefore:

$$k_1 \ x_1 = k_2 \ x_2 = F$$

$$x_1 = \frac{F}{k_1} \qquad x_2 = \frac{F}{k_2}$$

Series



- Since the total displacement is $x = x_1 + x_2$, and we have $F = k_{eq} x$

$$x = x_1 + x_2 \Rightarrow \frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2}$$

Translational Spring

$$\frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2}$$

- Then we can obtain

$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}$$

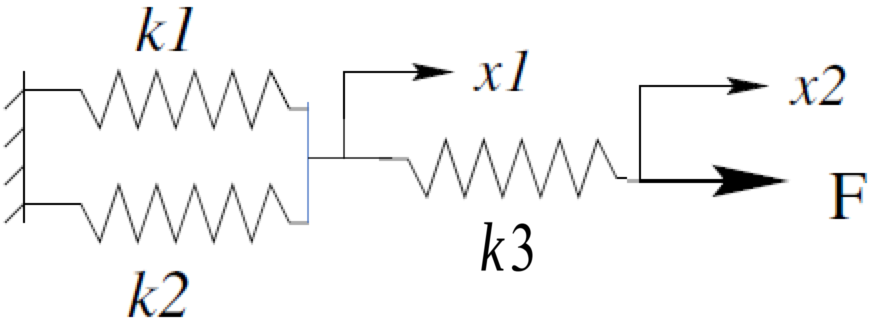
- If *n* springs are connected in series then:

$$k_{eq} = \frac{k_1 k_2 \cdots k_n}{k_1 + k_2 + \cdots + k_n}$$

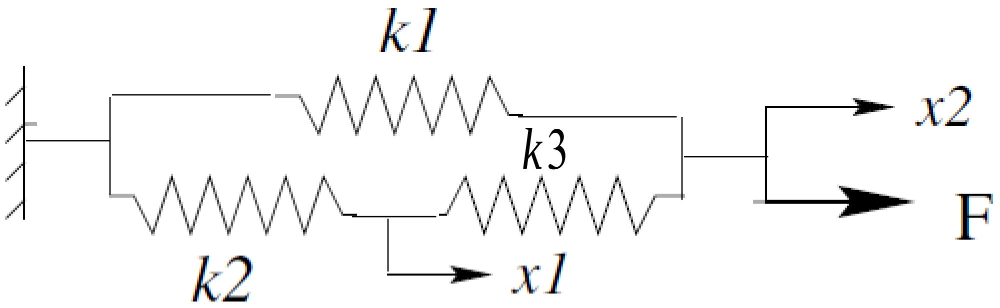
Homework: Translational Spring

- Obtain the equivalent stiffness for the following spring networks.

i)



ii)



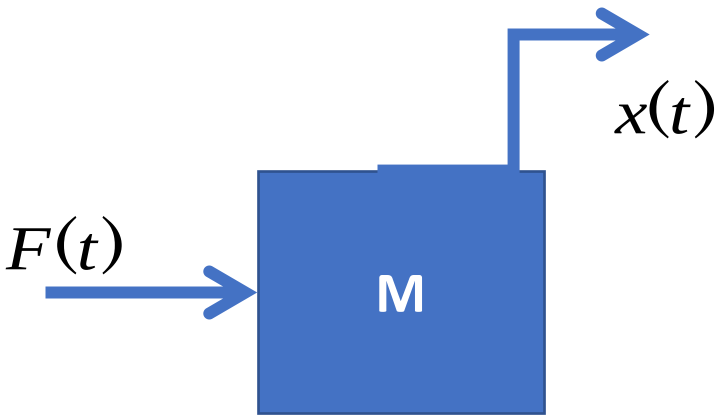
Translational Mass

- Translational Mass is an inertia element.
- A mechanical system without mass does not exist.
- If a force *F* is applied to a mass and it is displaced to *x* meters then the relation b/w force and displacements is given by Newton's law.

$$F = M\ddot{x}$$

Translational Mass

ii)

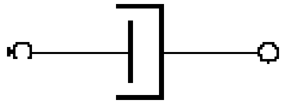


Translational Damper

- When the viscosity or drag is not negligible in a system, we often model them with the damping force.

- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

iii)



Common Uses of Dashpots

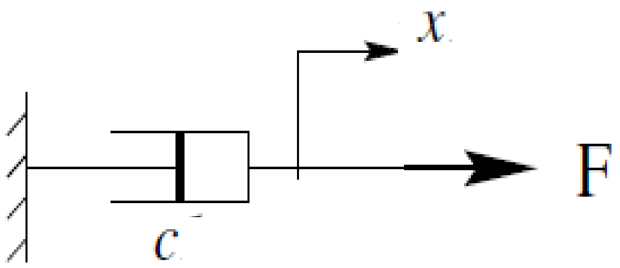
Door Stoppers

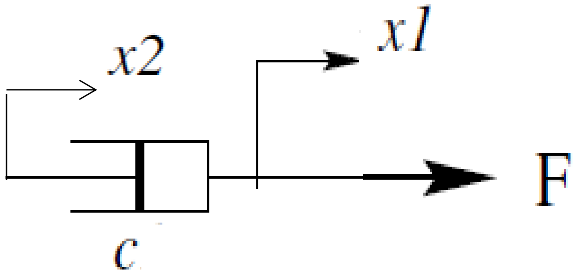
Vehicle Suspension

Bridge Suspension

Flyover Suspension

Translational Damper





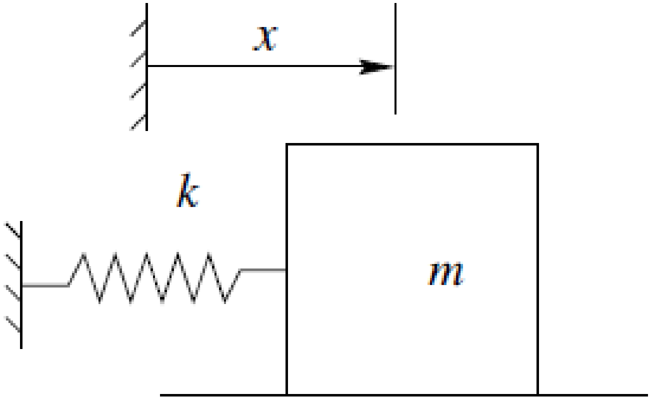
$$F = C\ \dot{x}$$

$$F = C\ (\dot{x}_1 - \dot{x}_2)$$

- Where C is damping coefficient ($N/m.\ s^{-1}$).

Modeling a simple Translational System

- **Example1 without force:** Consider a simple horizontal spring-mass system on a frictionless surface, as shown in figure below.



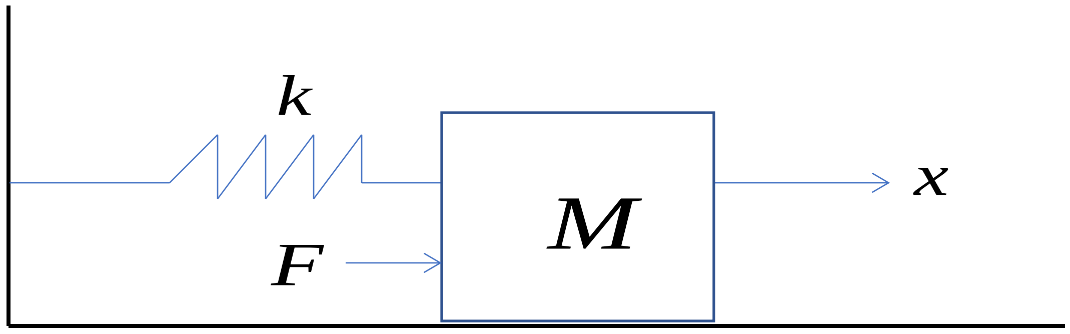
$$m\,\ddot{x} = -kx$$

or

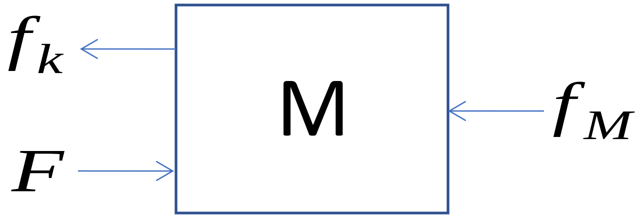
$$m\,\ddot{x} + kx = 0$$

Example 2 with external force:

- Consider the following system (friction is negligible)

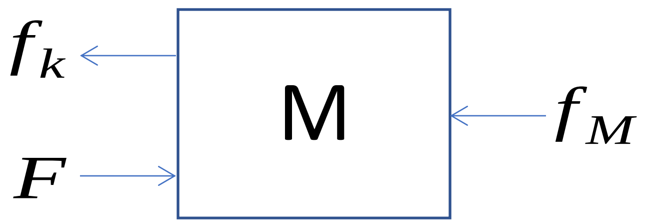


- Free Body Diagram



- Where f_k and f_M are force applied by the spring and inertial force respectively.

Example 2 with external force:



$$F = f_k + f_M$$

- Then the differential equation of the system is:

$$F = M\ddot{x} + kx$$

- Taking the Laplace Transform of both sides and ignoring initial conditions we get

$$F(s) = Ms^2X(s) + kX(s)$$

Example2 with external force:

$$F(s) = Ms^2X(s) + kX(s)$$

- The transfer function of the system is

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + k}$$

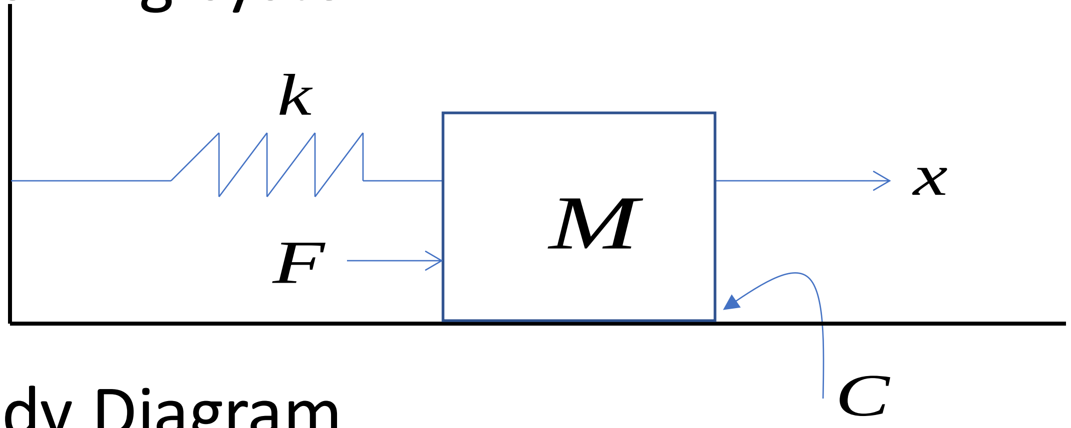
- if

$M = 1000kg$
$k = 2000Nm^{-1}$

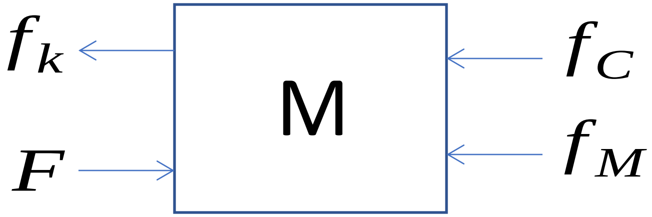
$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + 2}$$

Example 3

- Consider the following system



- Free Body Diagram



$$F = f_k + f_M + f_C$$

Example 3

Differential equation of the system is:

$$F = f_M + f_c + f_k$$

$$F = M\ddot{x} + C\dot{x} + kx$$

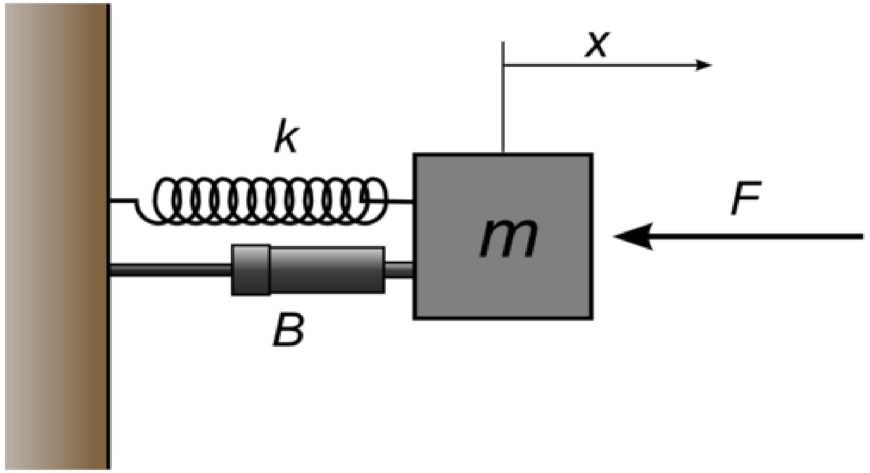
Taking the Laplace Transform of both sides and ignoring Initial conditions we get

$$F(s) = M\,s^2X(s) + CsX(s) + kX(s)$$

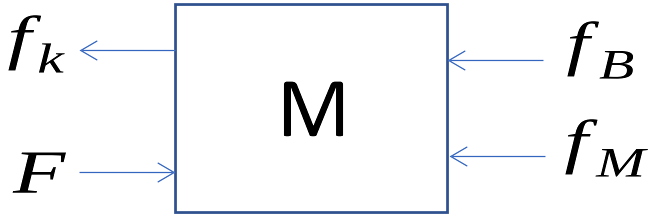
$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Cs + k}$$

Example 4

- Consider the following system



- Free Body Diagram (same as example-3)

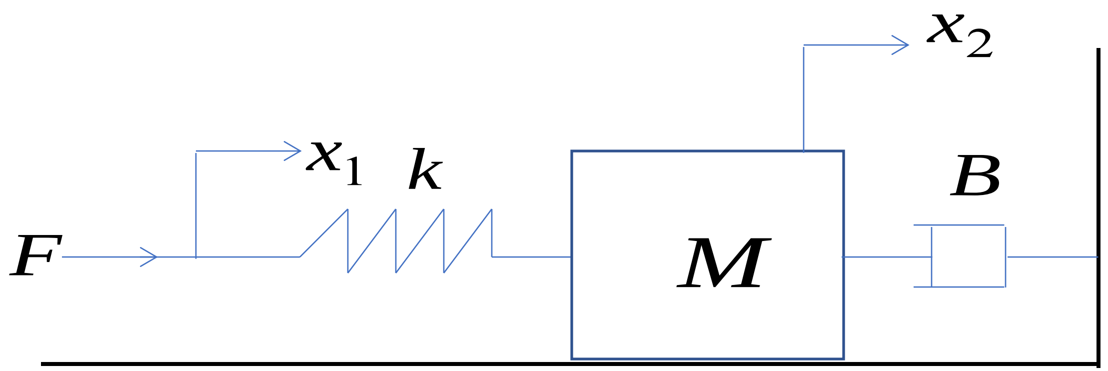


$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}$$

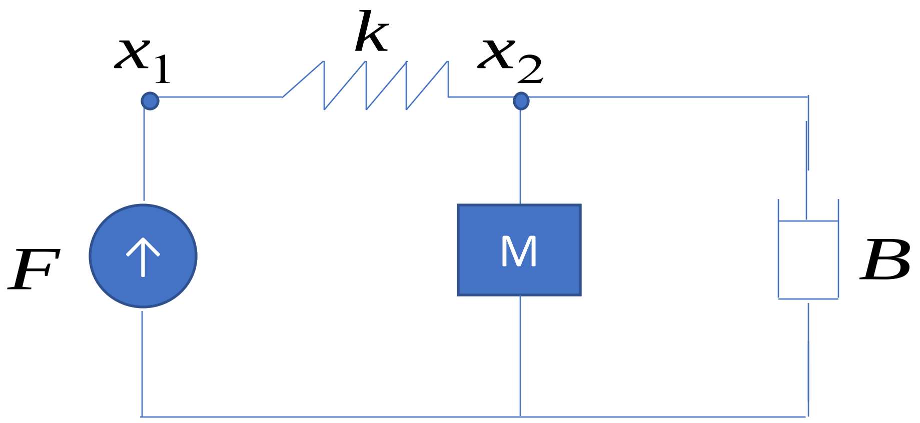
$$F = f_k + f_M + f_B$$

Example 5

- Consider the following system

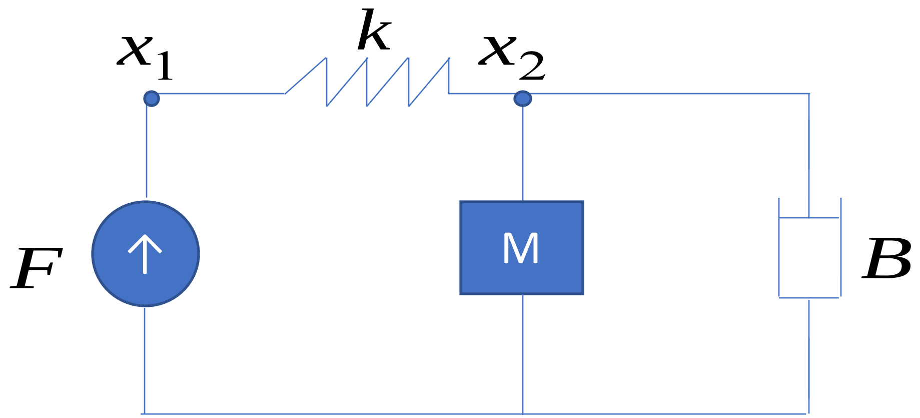


- Mechanical Network



Example 5

- Mechanical Network



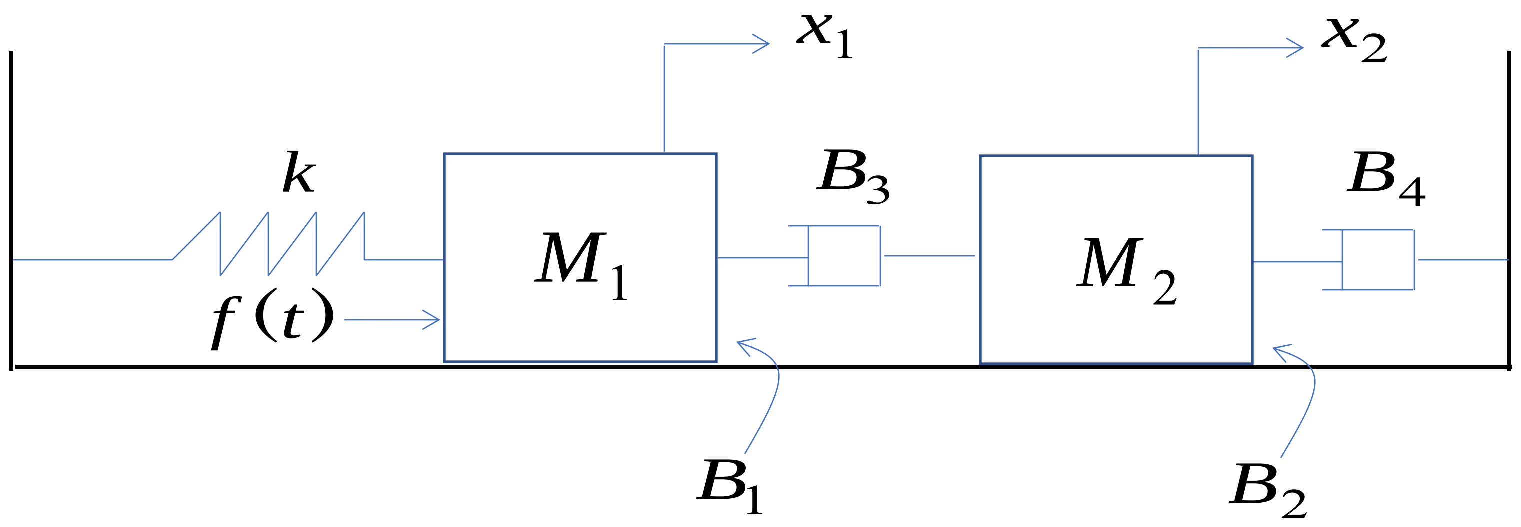
At node x_1

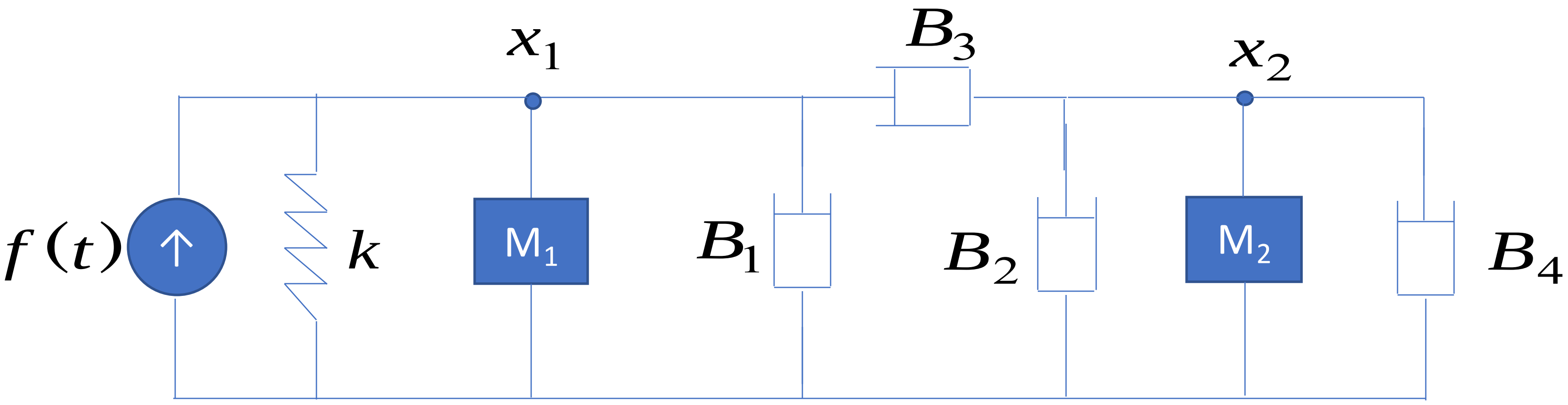
$$F = k(x_1 - x_2)$$

At node x_2

$$0 = k(x_2 - x_1) + M\ddot{x}_2 + B\dot{x}_2$$

Example 6





Example 7

- Find the transfer function of the mechanical translational system given in Figure-1.

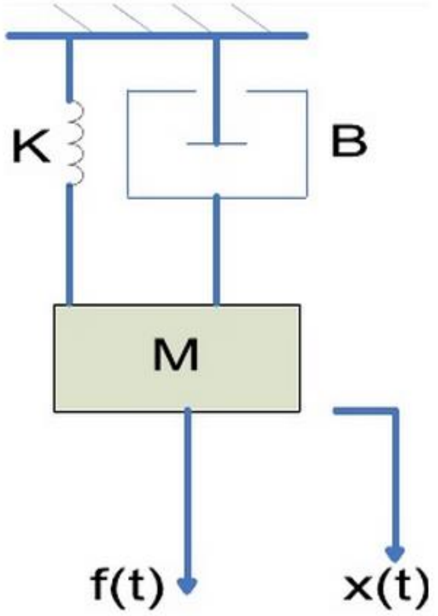
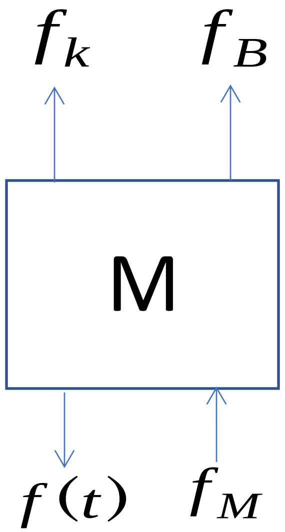


Figure-1

Free Body Diagram

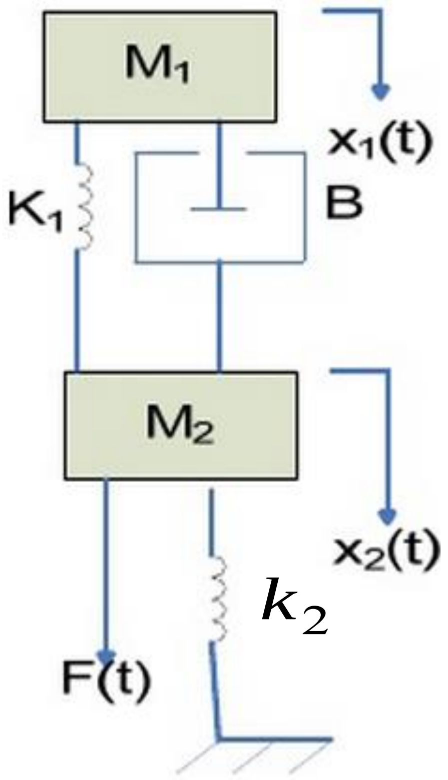


$$f(t) = f_k + f_M + f_B$$

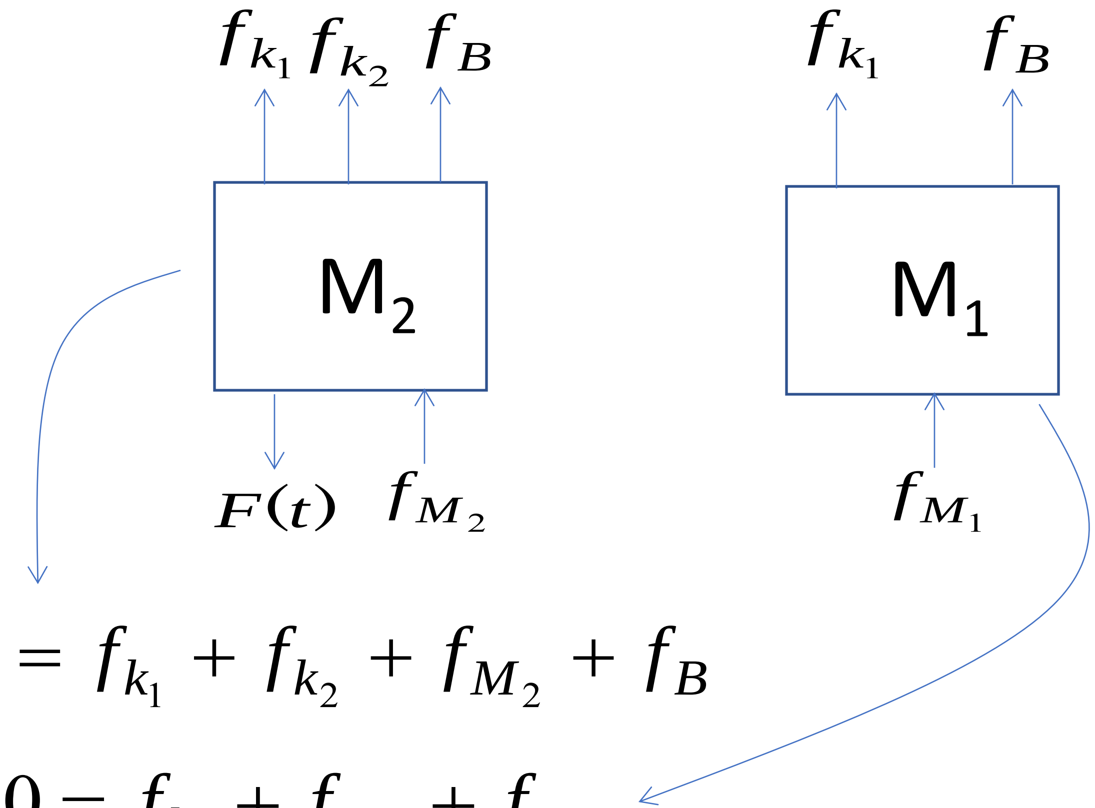
$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}$$

Example 8

- Find the transfer function $X_2(s)/F(s)$ of the following system.



Free Body Diagram

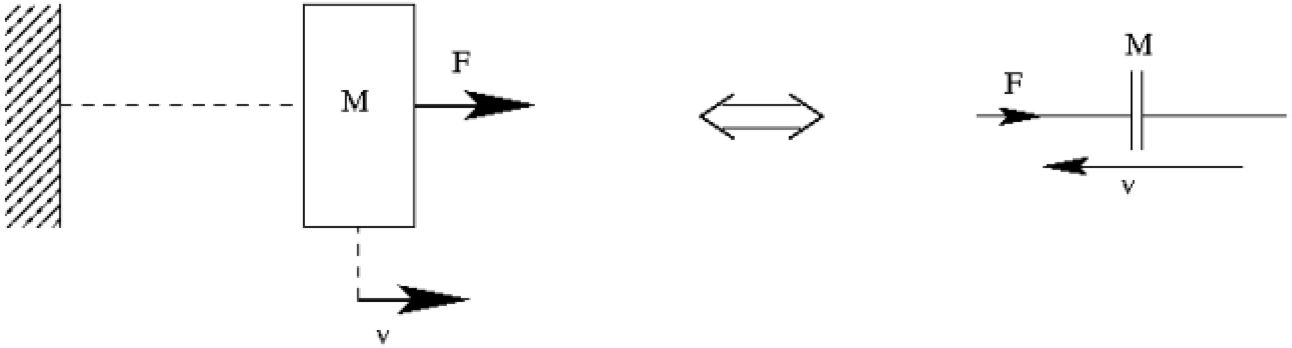


$$F(t) = f_{k_1} + f_{k_2} + f_{M_2} + f_B$$

$$0 = f_{k_1} + f_{M_1} + f_B$$

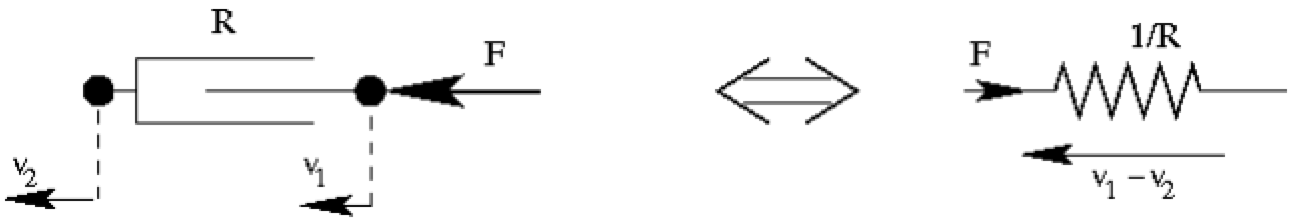
Analogous Electrical and Mechanical Systems

Mass



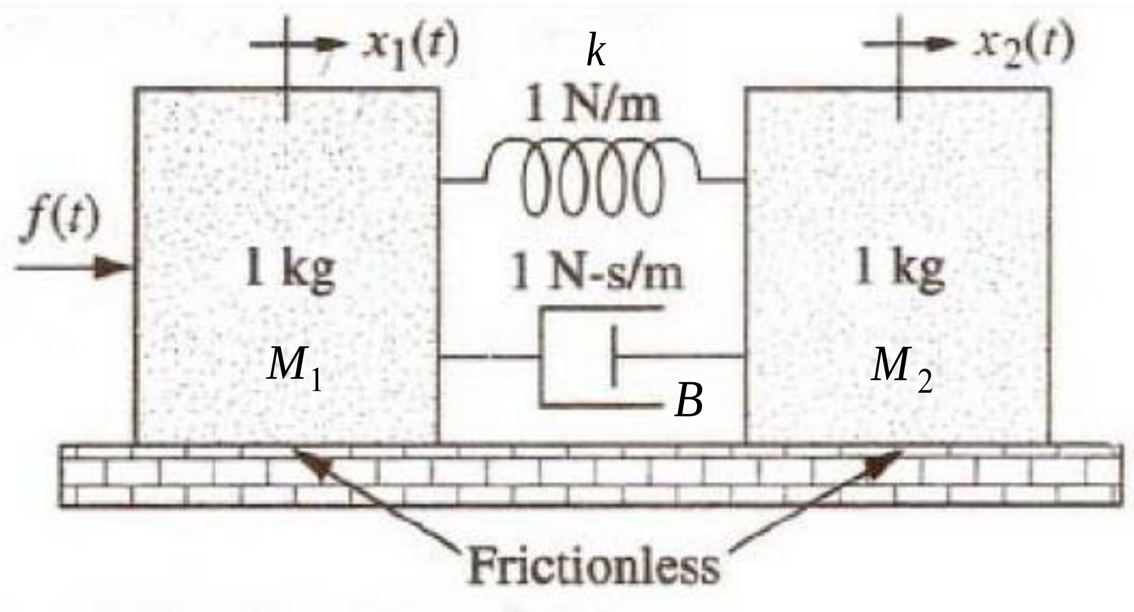
Spring

Damper



Homework

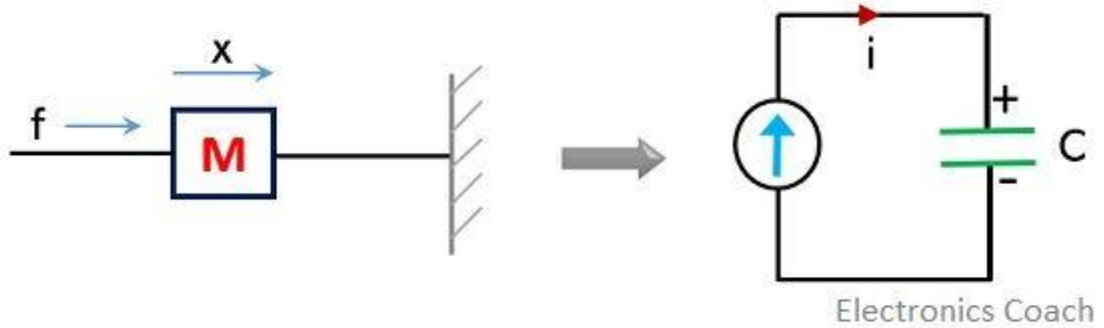
- Find the transfer function $X_2(s)/F(s)$ of the following system.

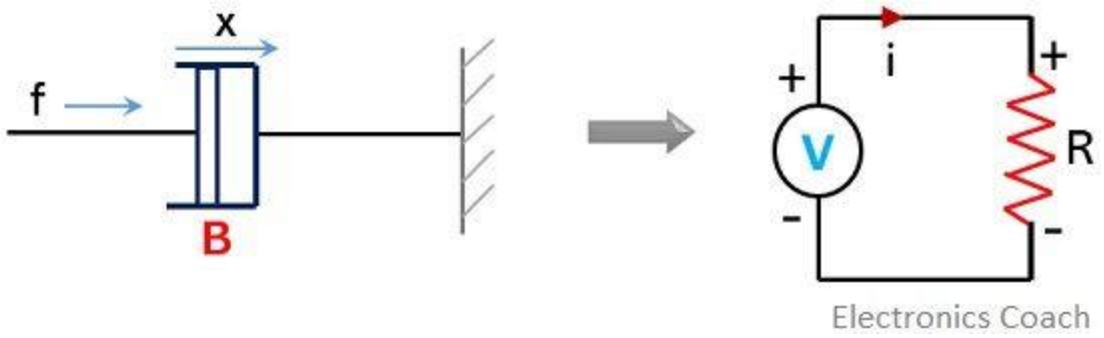


Analogous Electrical and Mechanical Systems

Background

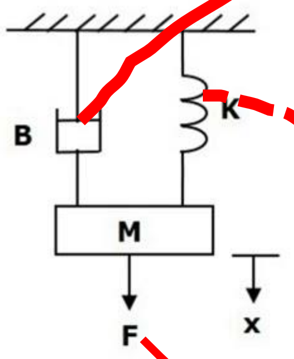
It is possible to make electrical and mechanical systems using analogs. An analogous electrical and mechanical system will have differential equations of the same form. There are two analogs that are used to go between electrical and mechanical systems. The analogous quantities are given below.



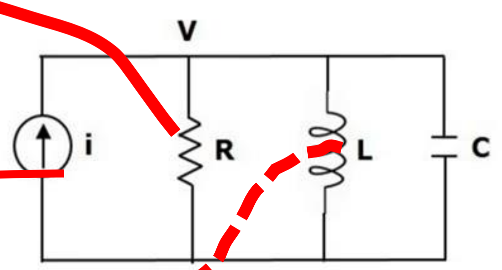


Analogous Electrical and Mechanical Systems

Force – Current Analogy



$$F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$



Applying KCL

$$i = \frac{v}{R} + \frac{1}{L}\int v dt + C\frac{dv}{dt}$$

But, $v = \frac{d\phi}{dt}$

$$i = \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi + C\frac{d^2\phi}{dt^2}$$

$$i = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi$$

Mechanical System	Electrical System
Force	Current
Mass (M)	Capacitor (C)
Dashpot (B)	Reciprocal of Resistor (1/R)
Spring (K)	Reciprocal of Inductor (1/L)
Displacement (x)	Flux (Φ)

Example:

1- Choose displacements as nodes to draw mechanical Network.

2- Connect all masse between ground and corresponding nodes.

3- Fit the mechanical elements interconnecting node in the network.

Analogous Electrical and Mechanical Systems

Choose displacements x1,x2,x3 as nodes to draw mechanical Network.

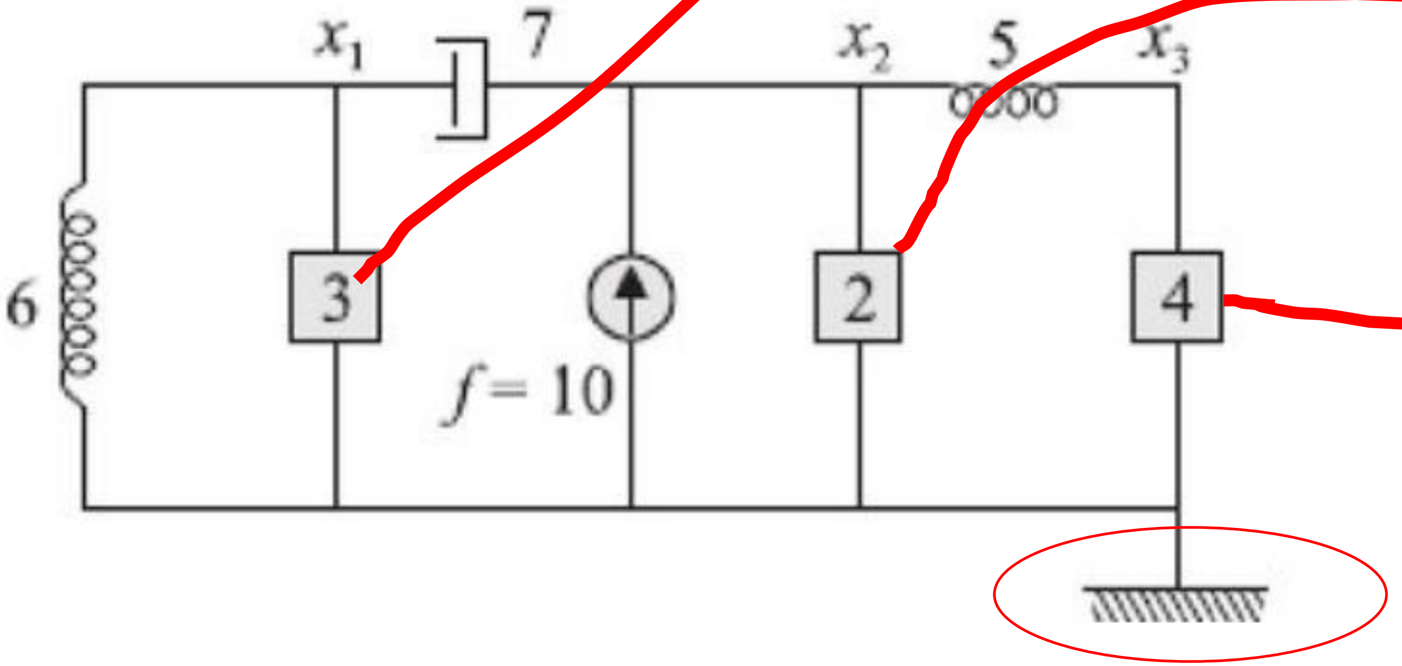
Example:

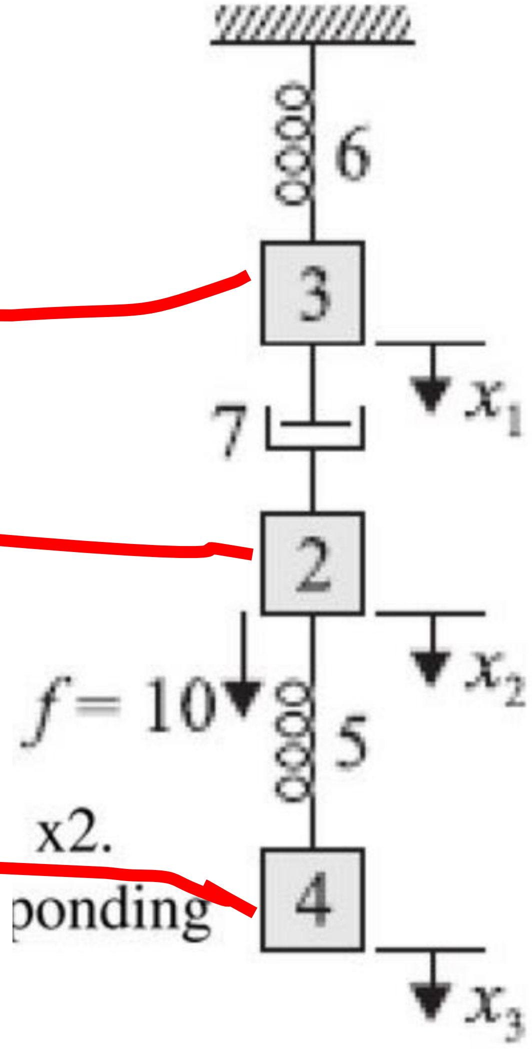
1- Choose displacements x1,x2,x3 as nodes to draw mechanical Network.

2- Connect all masses between ground and corresponding nodes.

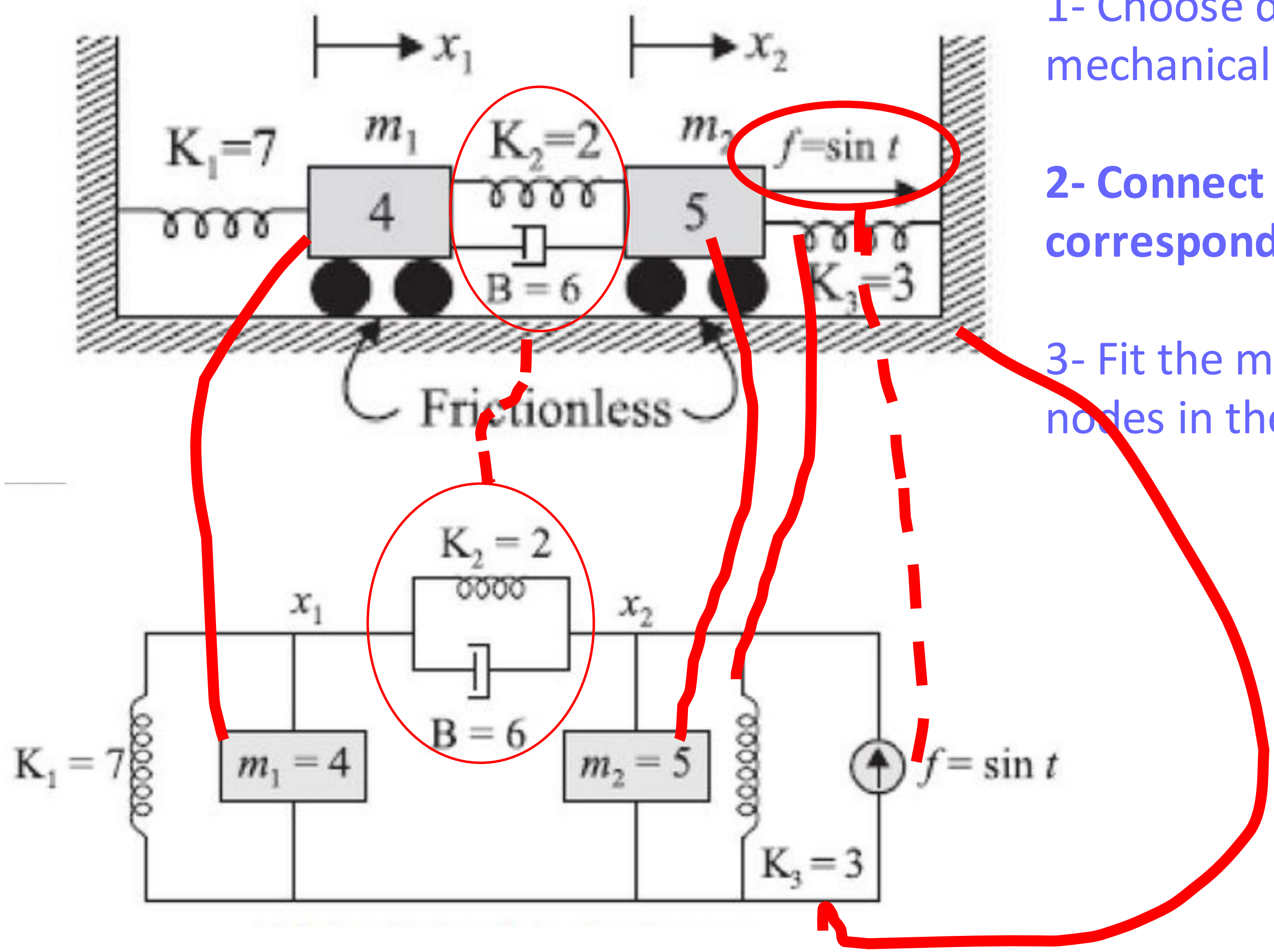
3- Fit the mechanical elements interconnecting two nodes in the network.

Ex: damper B= 7 between x1 and x2. Connect remaining elements between ground and corresponding





Example: Develop differential equation model for mechanical system



1- Choose displacements x1,x2 as nodes to draw mechanical Network.

2- Connect all masses between ground and corresponding nodes.

3- Fit the mechanical elements interconnecting two nodes in the network.

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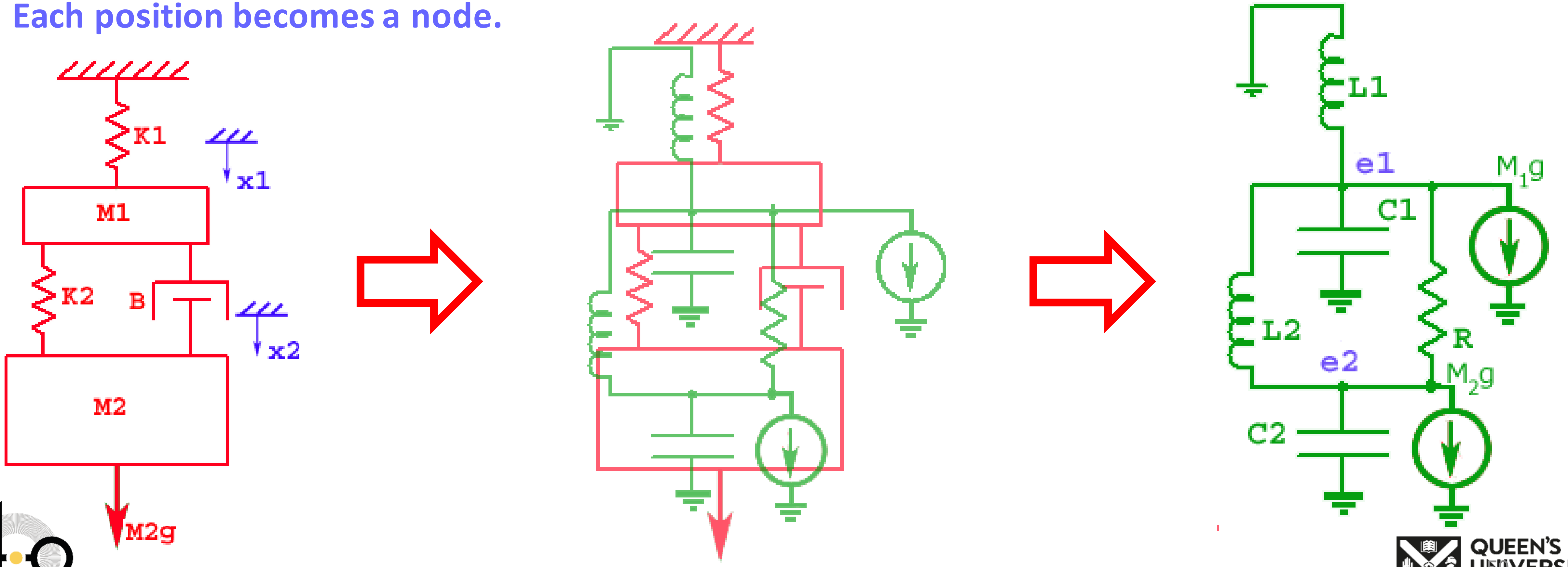
Example1: Conversion from Mechanical 1 to Electrical -- Visual Method

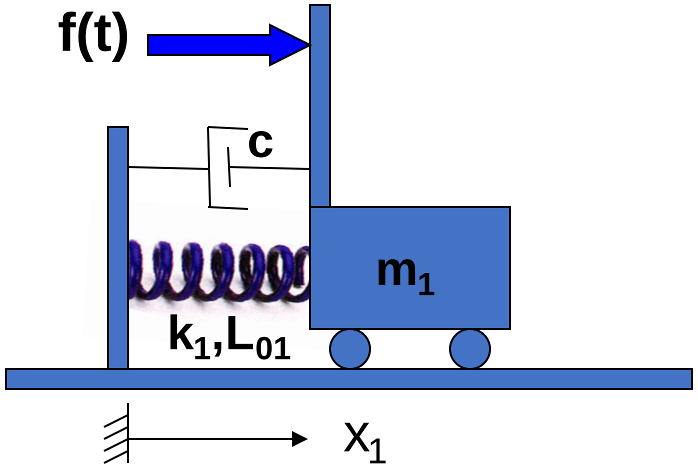
Draw over circuit, replacing mechanical elements with their analogs;

- force generators by current sources, input velocities by voltage sources,
- friction elements by resistors,
- springs by inductors,
- and masses by capacitors (which are grounded).
- Each position becomes a node.



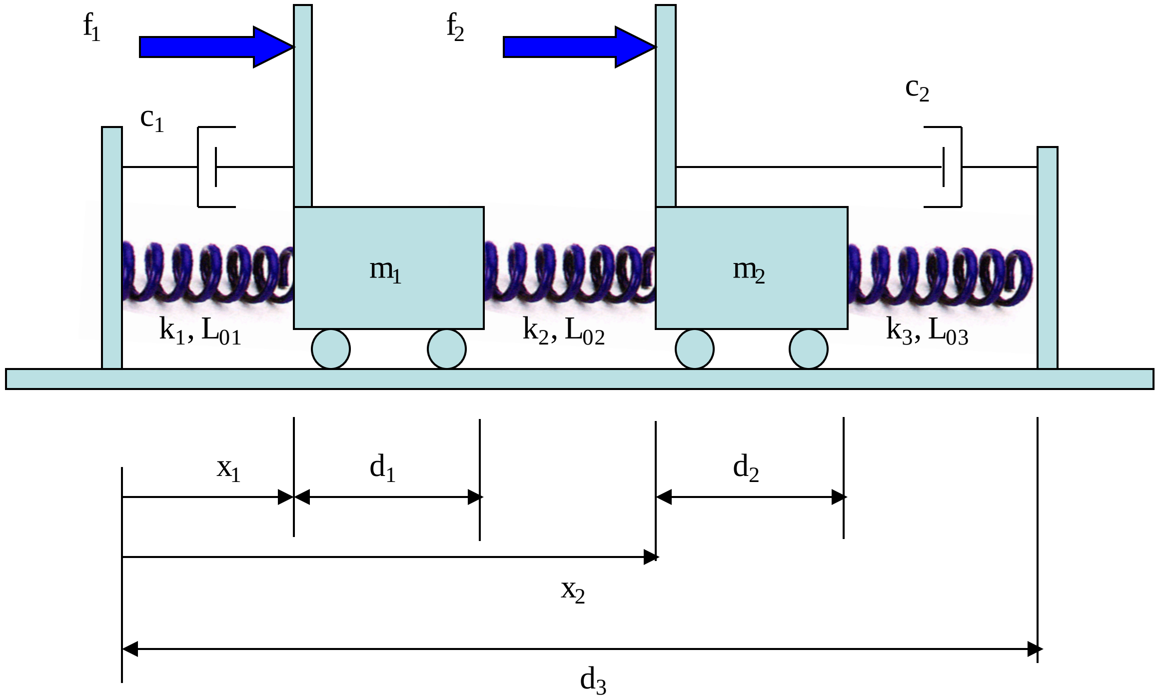
Homework

How to model the system?



Homework

How to model the system?



d1 = d2 = 15 cm
d3 = 80 cm
m1 = 2.5 kg
m2 = 5.5 kg
c1 = 8 N sec/m
c2 = 10 N sec/m
k1 = 3 N/cm ; L01 = 6 cm
k2 = 5 N/cm ; L02 = 8 cm
k3 = 4 N/cm ; L03 = 10 cm

Cars 1 and 2 start at rest at their static equilibrium position.

A constant force f_1 = 20N and f_2 = -5 N .

Obtain the motion response of the two cars, i.e. obtain $x_1(t)$ and $x_2(t)$.

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Thank You For Your Attention!

Any Question?