Mechanical Systems/ Modelling of Mechanical Systems- MEC100x – Lectures 3_2

Energy, Power and Intelligent Control

School of Electronics, Electrical Engineering and Computer Science

Ashby Building

Queen’s University Belfast

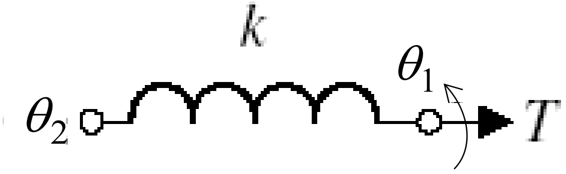
Aims

1. Modelling dynamic systems
2. Second-order systems
3. Balancing of rotating masses/Rotational-translational systems
4. Natural frequency
5. Compliance of dynamic elements
6. Transmissibility, transfer of motion through the support of a dynamic system

Rotational Mechanical Systems

Basic Elements of Rotational Mechanical Systems

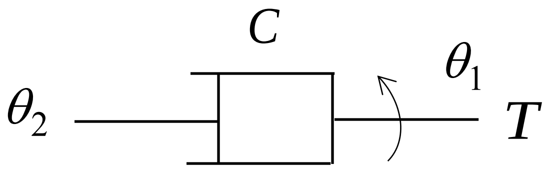
Rotational Spring



$$T = k(\theta_1 - \theta_2)$$

Basic Elements of Rotational Mechanical Systems

Rotational Damper

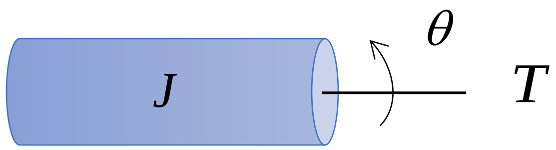


$$T = C(\dot{\theta}_1 - \dot{\theta}_2)$$

$$T = C(w_1 - w_2)$$

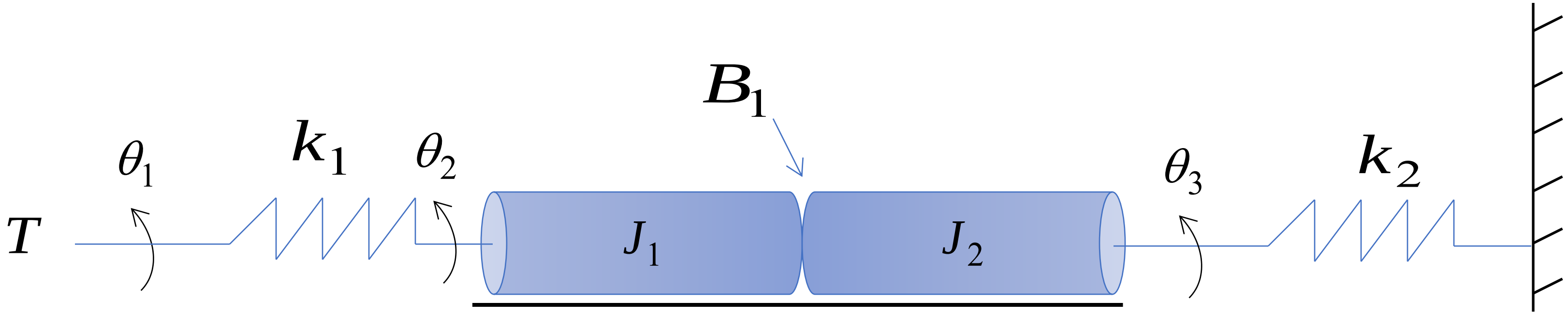
Basic Elements of Rotational Mechanical Systems

Moment of Inertia

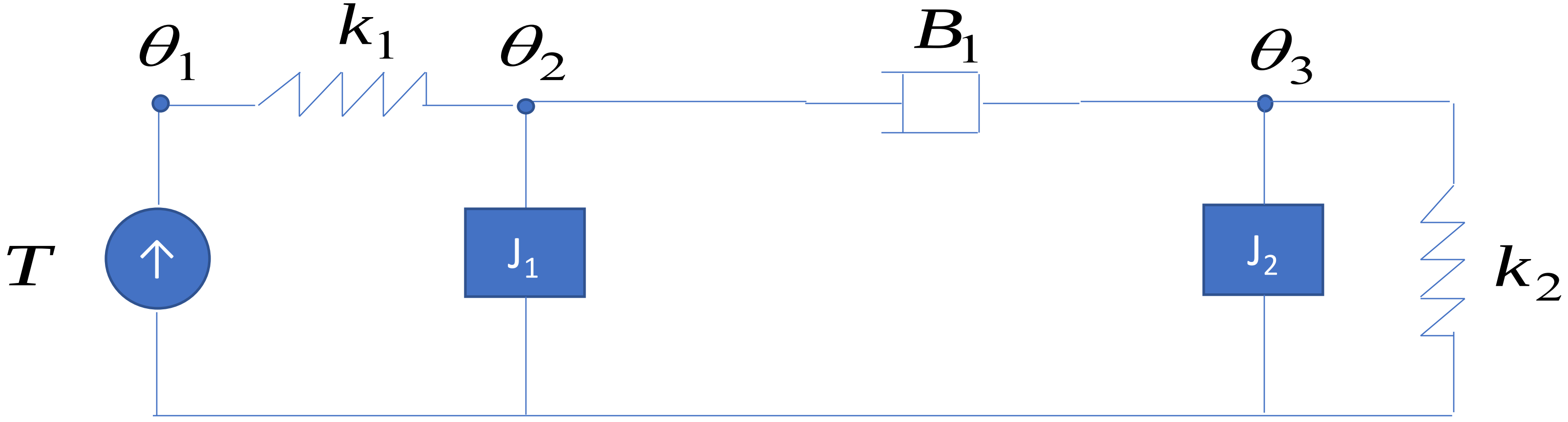


$$T = J\ddot{\theta}$$

Example-1

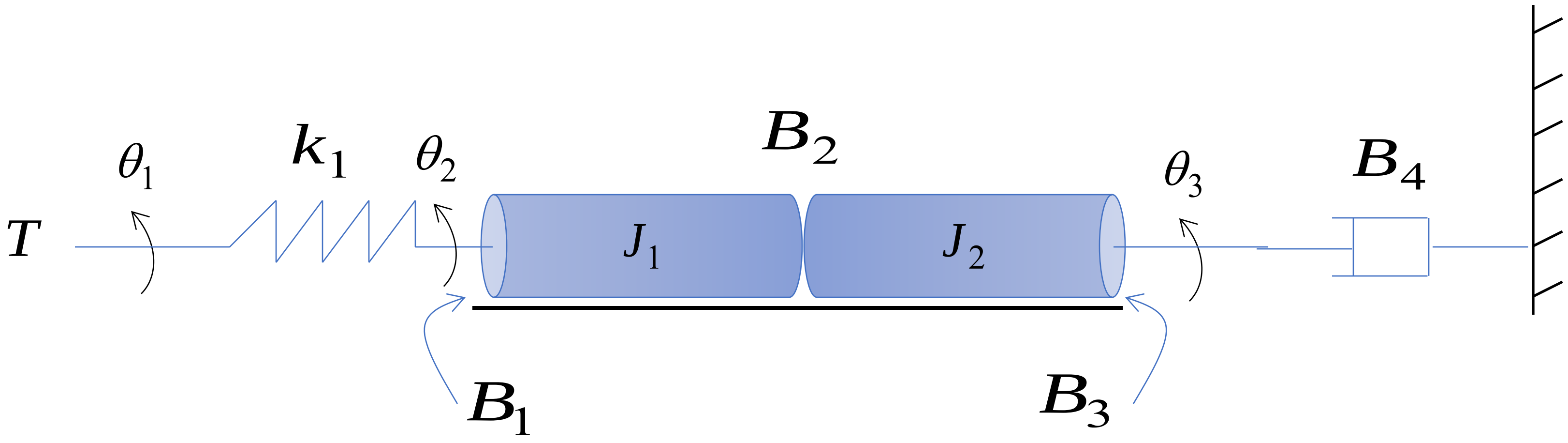


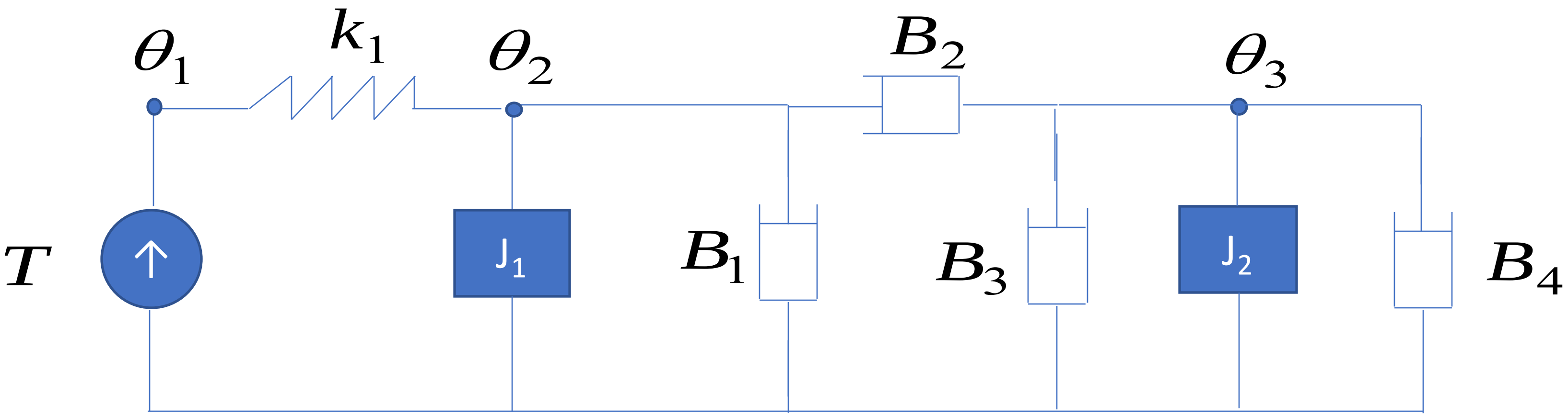
B_1
θ_1
k_1
θ_2
θ_3
k_2
T
J_1
J_2



θ_1
k_1
θ_2
B_1
θ_3
T
J_1
J_2
k_2

Example-2





Contents

- Stiffness in Precision Engineering
- Compliance of (a combination of) dynamic elements
- Dynamic modelling of damped mass-spring systems.
- Transmissibility
- Coupled mass-spring systems
- Standard mechanical frequency responses

What is a realistic low stiffness value in a mechanical connection?

1. 1
2. 10
3. 100
4. 1000
5. 10000
6. 100000
7. 1000000

Values in N/m

What is a realistic high stiffness value in a mechanical connection?

1. 10^5
2. 10^6
3. 10^7
4. 10^8
5. 10^9
6. 10^{10}

Values in N/m

Stiffness of objects

	Well known objects
10^2 N/m	Soft pillow
10^4 N/m	Car suspension Soft couch
10^5 N/m	Table Bicycle
10^7 N/m	Office building
10^8 N/m	Concrete pillar
10^9 N/m	Steel train wheel on steel rail track

What is stiffness in precision machining?

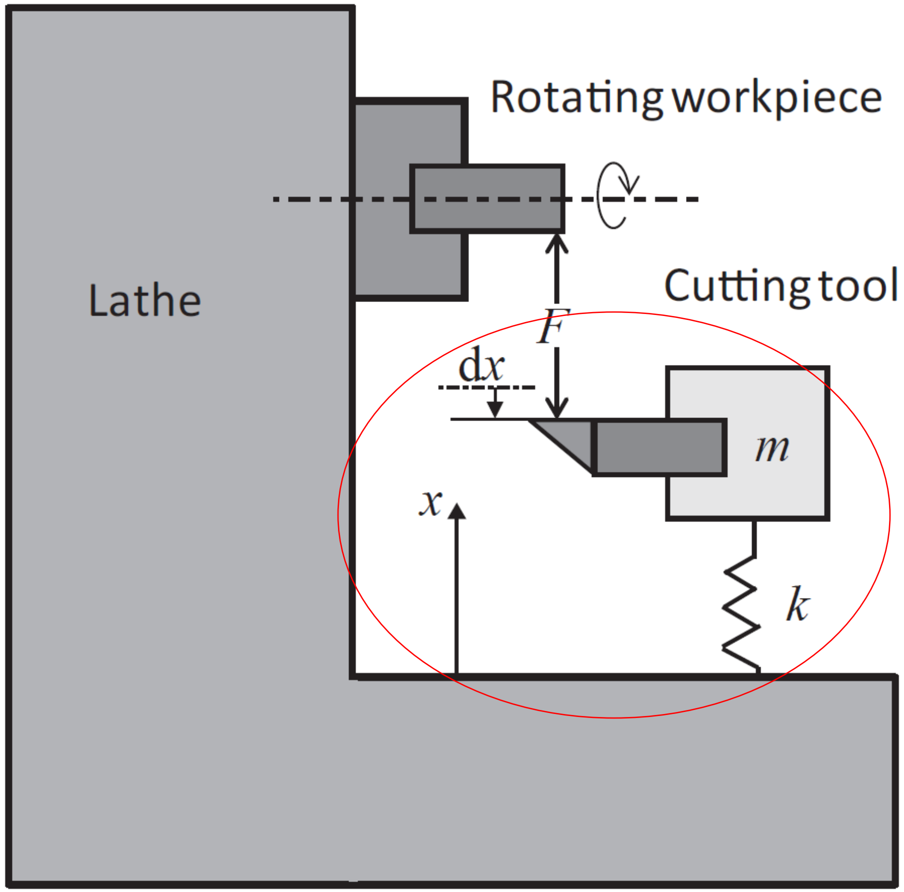
Hooke's law for force from spring:

$$F_s = -k\mathrm{d}x$$

"Hooke-Newton" law for external force:

$$F_r = F = k\mathrm{d}x$$

$$\mathrm{d}x = \frac{F}{k}$$

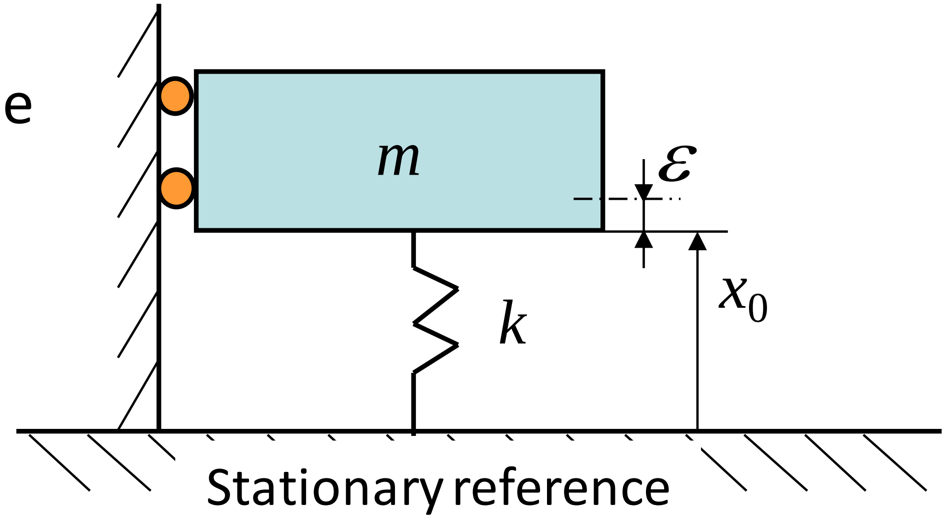


Where should you place the stiffness if possible?

➔ Take the shortest force loop

Natural frequency of the resonance of a **mass-spring system**

- At resonance the forces are in balance
- Deformation force (stiffness) plus acceleration force (mass) is zero.



$$F_a + F_d = m\frac{d^2x}{dt^2} + kx = 0 \Rightarrow m\frac{d^2x}{dt^2} = -kx$$

Fv=0

Fe=0

$$x = \hat{x}\sin(w_0 t) \Rightarrow$$

$$-m\hat{x}w_0^2\sin(w_0 t) = -k\hat{x}\sin(w_0 t) \Rightarrow w_0 = \sqrt{\frac{k}{m}}$$

Spring stiffness

mass

Natural frequency

The first natural frequency determines the sensitivity to harmonic vibrations!

$$F_a + F_d = m\frac{d^2x}{dt^2} + kxe = 0 \quad \Rightarrow \quad m\frac{d^2x}{dt^2} = -kxe$$

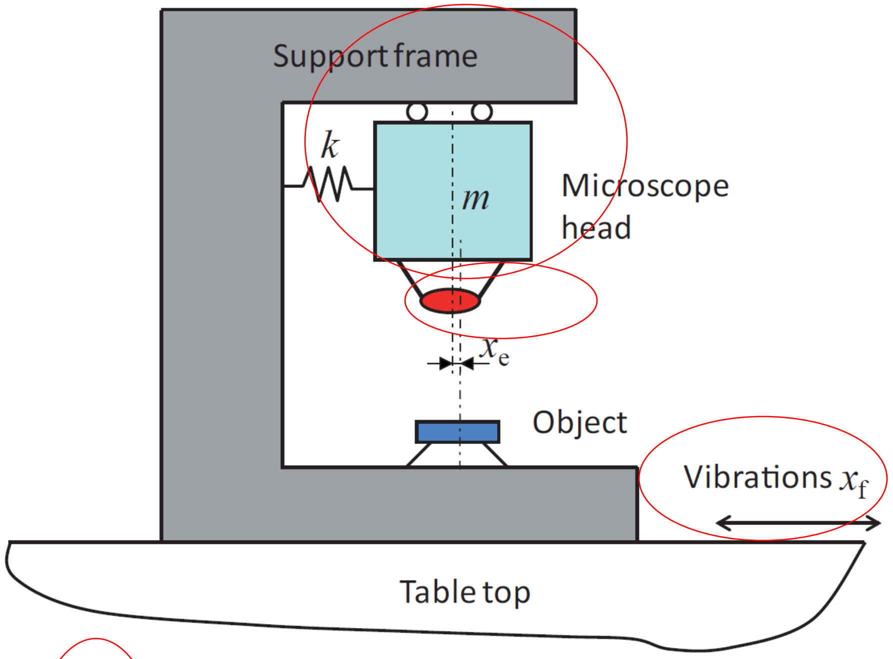
$$x = \hat{x}_f \sin(wt)$$

- The maximum force needed to follow the acceleration:

$$\hat{F} = m\hat{a} = m\hat{x}_f\omega^2$$

- The maximum error due to this force:

$$\hat{x}_e = \frac{\hat{F}}{k} = \frac{m\hat{x}_f\omega^2}{k} \rightarrow \frac{\hat{x}_e}{\hat{x}_f} = \frac{m\omega^2}{k}$$



- The natural frequency: $\frac{k}{m} = \omega_0^2$

- Which results in: $$\hat{x}_e = \hat{x}_f\frac{\omega^2}{\omega_0^2} \Rightarrow \frac{\omega_0}{\omega} = \frac{f_0}{f} = \sqrt{\frac{\hat{x}_f}{\hat{x}_e}} \Rightarrow f_0 \geq f\sqrt{\frac{\hat{x}_f}{\hat{x}_e}}$$

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Stiffness and compliance

- Stiffness is the ability of a system to withstand a force by minimising the resulting motion/deformation

- Compliance is the opposite of stiffness

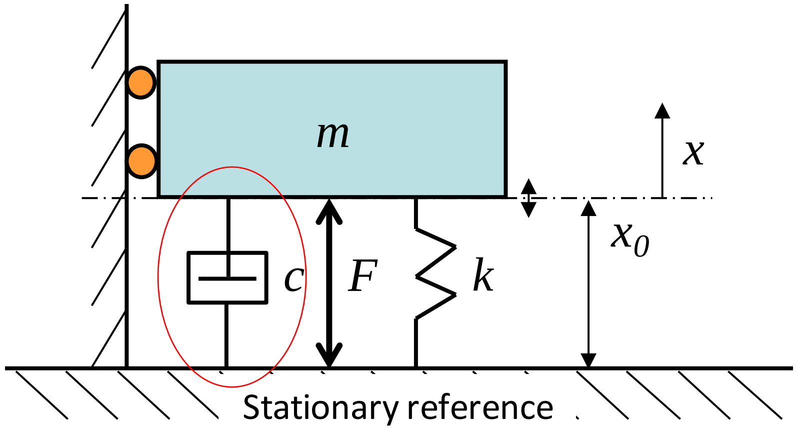
- Both can be real, in phase with a periodic force, or complex, dynamic, frequency dependent, 90^{o} out of phase with a periodic force.

Output/input

- **A spring has a real stiffness/compliance**:

$$C_s = \frac{x}{F} = \frac{1}{k}$$

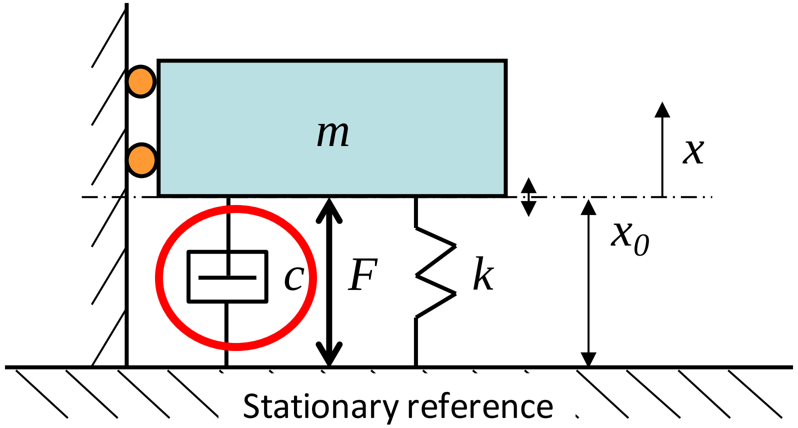
Compliance of (a combination of) dynamic elements



$$C_s = \frac{x}{F} = \frac{1}{k}$$

- k = stiffness of **the spring**
- c = damping coefficient of the damper
- m = **mass** of the body

Stiffness and compliance of a damper



Fv= c .(V(velocity)=(dx/dt)

Fv(s)= c. S. X(s)

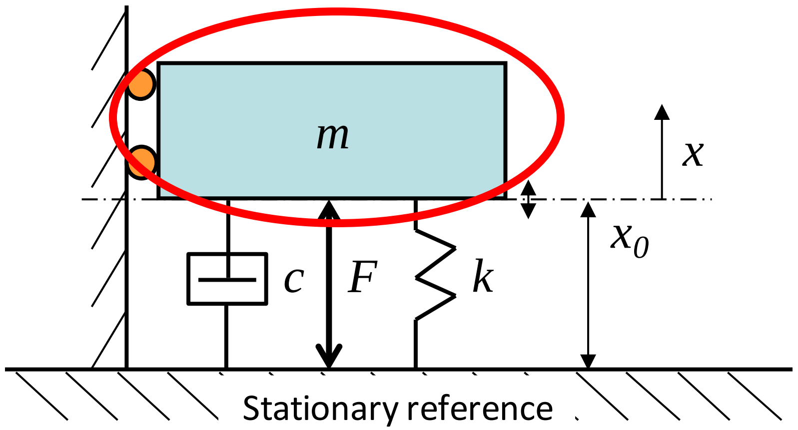
$$F(t) = c\frac{dx}{dt}, \quad F(s) = \mathcal{L}\{F(t)\} = scx$$

$$F(\omega) = \mathcal{F}\{F(t)\} = jc\omega x$$

S=jw

$$C_d(\omega) = \frac{1}{k_d(\omega)} = \frac{x}{F} = \frac{1}{jc\omega}$$

Stiffness and compliance of a mass body



F=M(mass) . a(acceleration)=d2x/dt2

S^2 . x

$$F(t) = m\frac{d^2x}{dt^2}, \quad F(s) = \mathcal{L}\{F(t)\} = ms^2x$$

S=jw

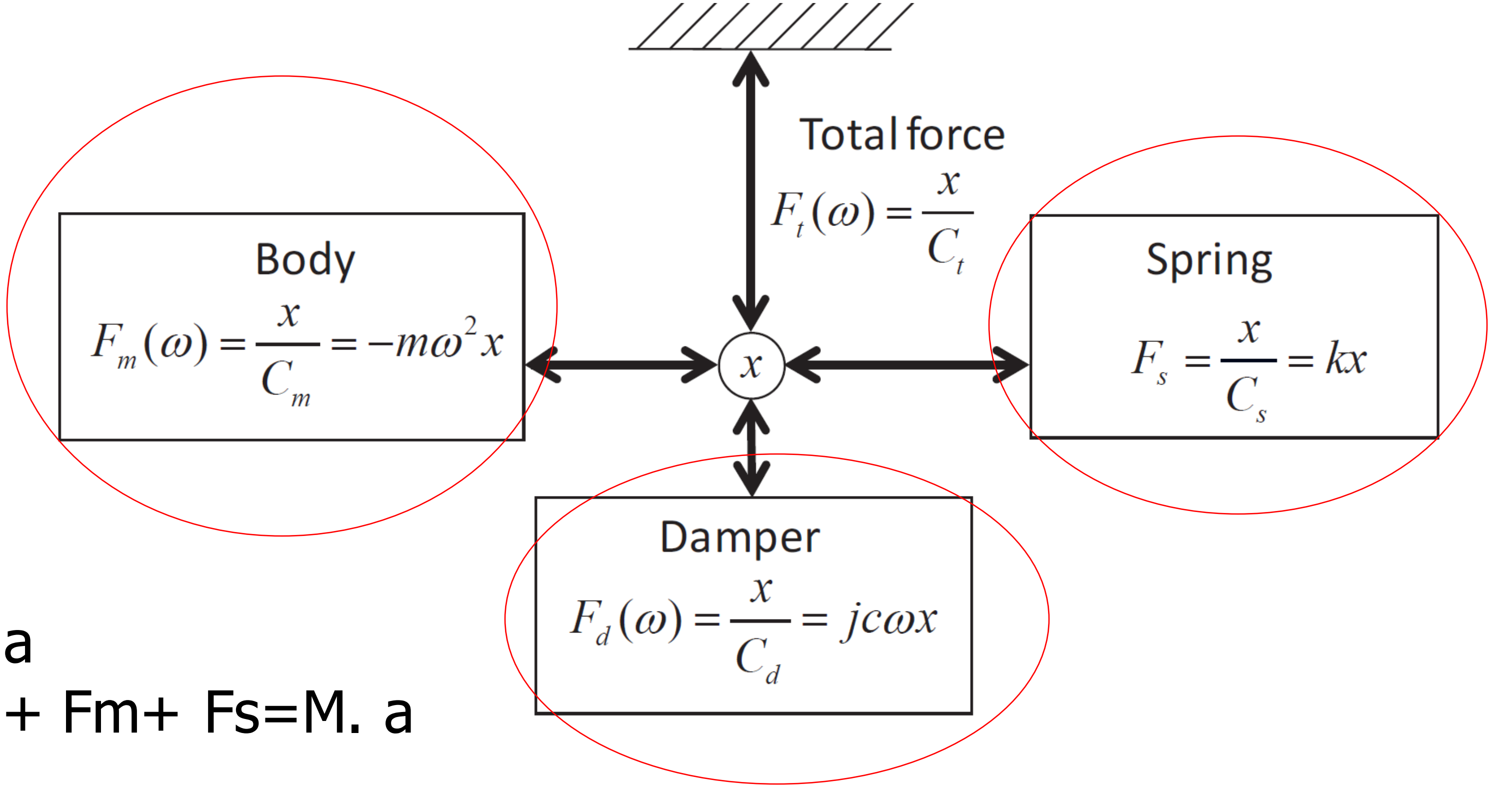
$$F(\omega) = \mathcal{F}\{F(t)\} = -m\omega^2 x$$

$$C_m(\omega) = \frac{1}{k_m(\omega)} = \frac{x}{F} = -\frac{1}{m\omega^2}$$

Output/input

F=k. x → x/F=1/k

Combined Compliance of body, spring and damper



F= M. a

F= Fv + Fm+ Fs=M. a

$$F_t(\omega)=F_s+F_d(\omega)+F_m(\omega)=x\left(\frac{1}{C_s}+\frac{1}{C_d}+\frac{1}{C_m}\right)=\frac{x}{C_t}$$

$$C_t(\omega)=\frac{x}{F_t}(\omega)=\frac{1}{\frac{1}{C_s}+\frac{1}{C_d}+\frac{1}{C_m}}$$

Overview of the dynamic properties

Item	Spring	Damper	Body
Variable	k	c	m
External force	$F_s(\omega)=kx$	$F_d(\omega)=jc\omega x$	$F_b(\omega)=-m\omega^2 x$
(dynamic) Stiffness	$k_s(\omega)=\frac{F_s}{x}=k$	$k_d(\omega)=\frac{F_d}{x}(\omega)=jc\omega$	$k_m(\omega)=\frac{F_b}{x}(\omega)=-m\omega^2$
(dynamic) Compliance	$C_s(\omega)=\frac{x}{F_s}=\frac{1}{k}$	$C_d(\omega)=\frac{x}{F_d}(\omega)=\frac{1}{jc\omega}$	$C_m(\omega)=\frac{x}{F_b}(\omega)=-\frac{1}{m\omega^2}$

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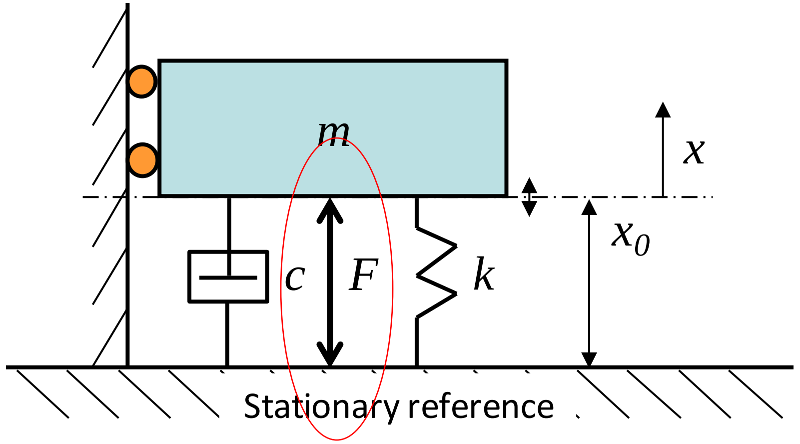
Start with second law of Newton

$F = m.a$



m.a c.V

$$F(t) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

Laplace gives:

$$F(s) = \mathrm{L}\{F(t)\} = x(ms^2 + cs + k)$$

Output/input= x/F= 1/ ms^2+ c. s + k

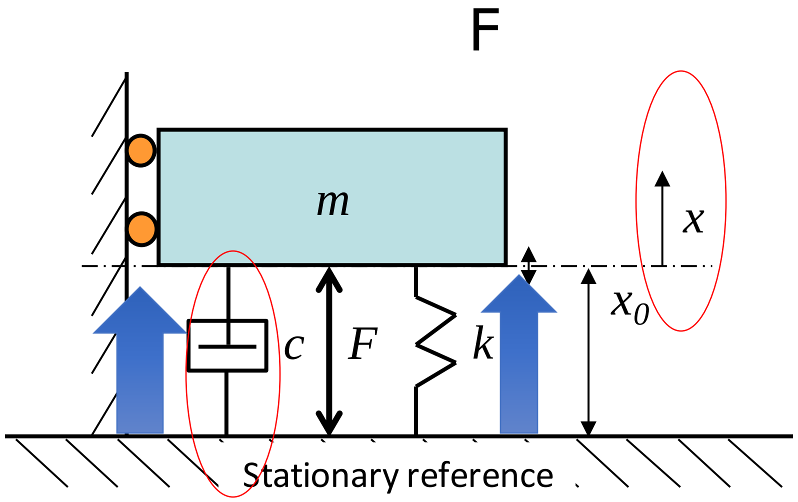
F= k. x

F= Ma

F= c. V= c. s. x

F= M. a + c.V + k. X

Start with second law of Newton **F=m.a**



X=output
F= input
F-Kx-C.v=m.a

$$F(t) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx)$$

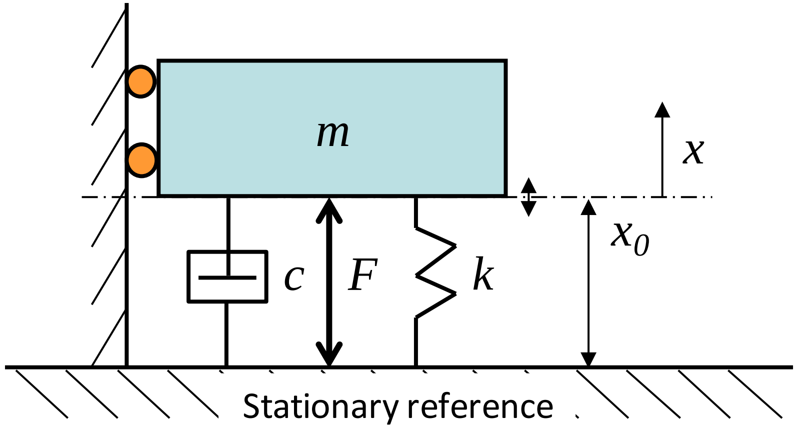
Laplace gives:

$$F(s) = L\{F(t)\} = x(ms^2 + cs + k)$$

$$C_t(s) = \frac{x_m}{F}(s) = \frac{1}{ms^2 + cs + k} = \frac{\frac{1}{k}}{\frac{m}{k}s^2 + \frac{cs}{k} + 1}$$

Output/input= x/F= 1/ m. s^2+ c. s + k= = 1/ m/k. s^2+ c/k. s +1

Start with second law of Newton **F=m.a**



$$F(t) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx)$$

Laplace gives:

$$F(s) = L\{F(t)\} = x(ms^2 + cs + k)$$

$$C_t(s) = \frac{x_m}{F}(s) = \frac{1}{ms^2 + cs + k} = \frac{\frac{1}{k}}{\frac{m}{k}s^2 + \frac{cs}{k} + 1}$$

With only positive imaginary terms (Fourier):

Natural frequency

$$\frac{1}{k} = C_s \quad \omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \frac{c}{2\sqrt{km}}$$

FRF

$$C_t(\omega) = F\{F(t)\} = \frac{x}{F}(\omega) = \frac{C_s}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}$$

C=damping , k=stiffness

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The damping ratio ζ is related to the pole location in the Laplace plane

$$s = \sigma + j\omega$$

Poles are those values of s where denominator of C_t is zero

$$C_t = \frac{x}{F} = \frac{\frac{1}{k}}{\frac{m}{k}s^2 + \frac{cs}{k} + 1} = \frac{C_s}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}$$

$$p_1 = -\sigma + j\omega_{d,n} \quad p_2 = -\sigma - j\omega_{d,n} \quad \sigma = \zeta\omega_0 \quad \omega_{d,n} = \omega_0\sqrt{1-\zeta^2}$$

Zeta=0 , C=0

If $c = 0$ then $\zeta = 0$ no damping!

$$C_t = \frac{C_s}{\frac{m}{k}s^2 + 1} \text{ and } p_1 = +j\omega_0 \text{ and } p_2 = -j\omega_0$$

w0=sqrt(k/m)

m/k S^2+1=0 →

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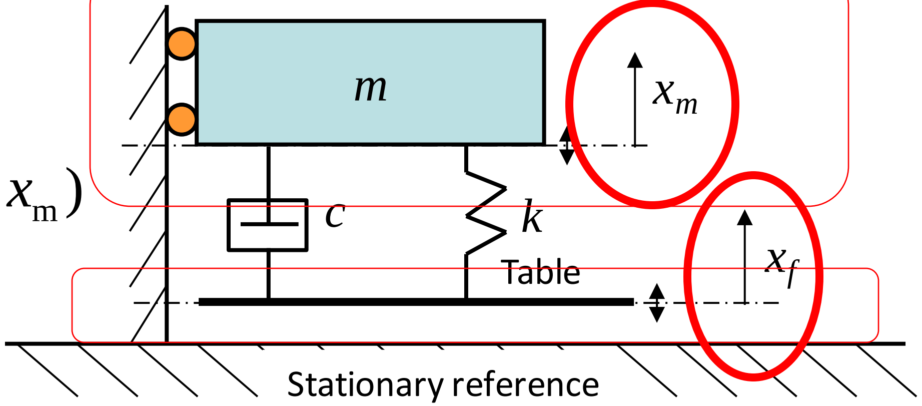
Transmissibility, transfer of motion through the support of a dynamic system

The force acting on the body equals

$$F_{t,m}(t) = m\frac{d^2 x_m}{dt^2} = c\frac{d(x_f - x_m)}{dt} + k(x_f - x_m)$$



$$\Rightarrow \quad x_m(ms^2 + cs + k) = x_f(cs + k)$$

$$\frac{x_m}{x_f} = \frac{cs + k}{ms^2 + cs + k} = \frac{\frac{cs}{k} + 1}{\frac{m}{k}s^2 + \frac{cs}{k} + 1}$$

Output=xm/input=xf

With: $\omega_0 = \sqrt{\frac{k}{m}}$ $\quad \zeta = \frac{c}{2\sqrt{km}}$ damping

Stiffness

$$\Rightarrow \quad \frac{x_m}{x_f} = \frac{2\zeta\frac{s}{\omega_0} + 1}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}$$

What means the additional dynamic term in the numerator.

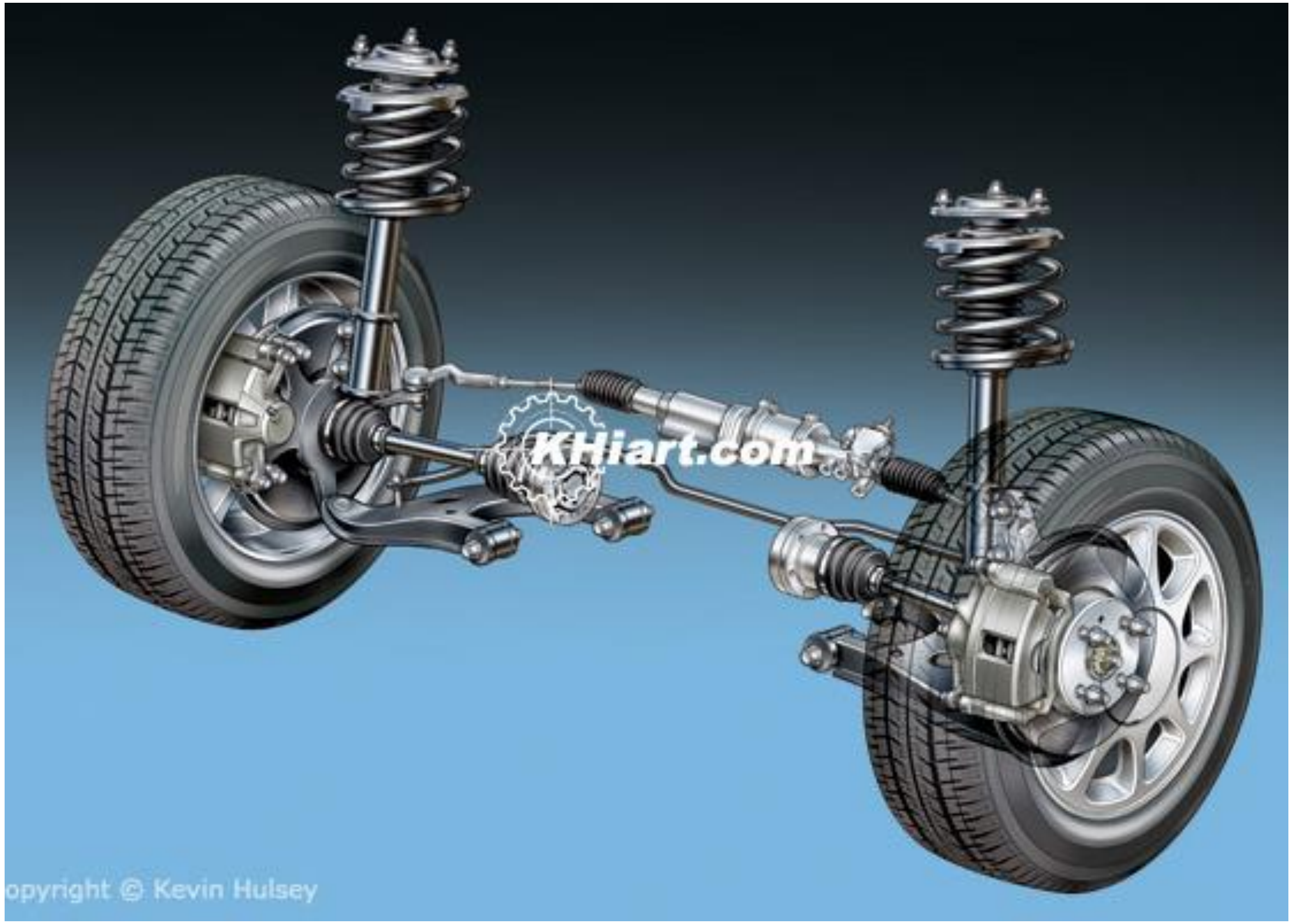
$$\frac{x}{F} = \frac{C_s}{\frac{s^2}{\omega_0^2} + 2\zeta \frac{s}{\omega_0} + 1}$$

Compliance

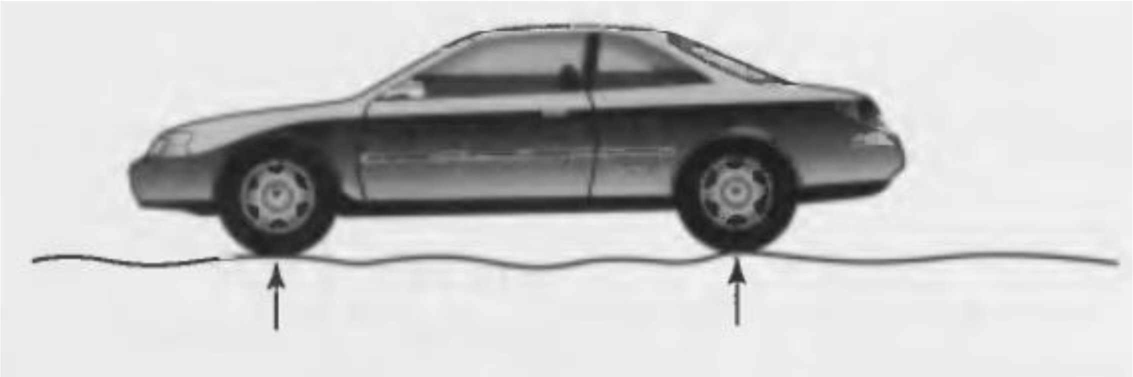
$$\frac{x_m}{x_f} = \frac{2\zeta \frac{s}{\omega_0} + 1}{\frac{s^2}{\omega_0^2} + 2\zeta \frac{s}{\omega_0} + 1}$$

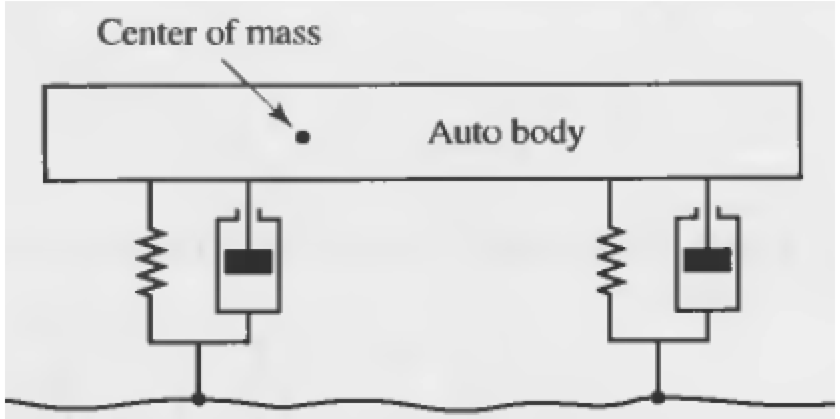
Transmissibility

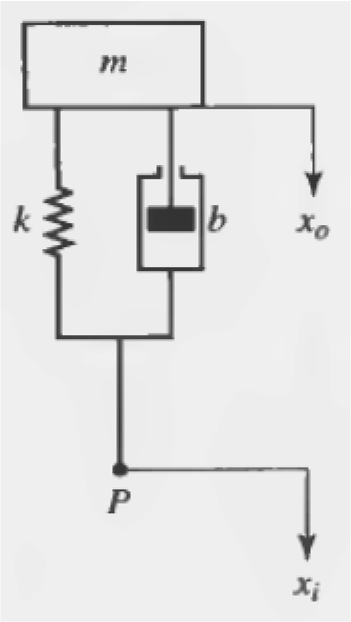
Example: Automobile Suspension



Automobile Suspension



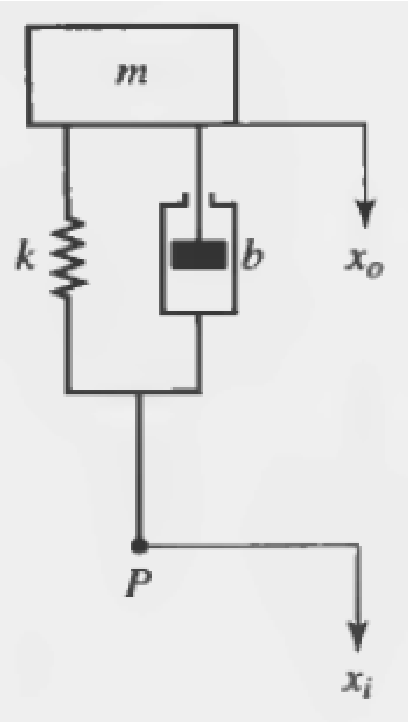




Automobile Suspension

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0 \qquad \text{(eq .1)}$$

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i \qquad \text{eq. 2}$$



Taking Laplace Transform of the equation (2)

$$ms^2 X_o(s) + bsX_o(s) + kX_o(s) = bsX_i(s) + kX_i(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Active controlled vehicle suspension

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Comfort of a car depends mainly on:

A. Stiffness of the suspension
B. Damping of the suspension
C. Stiffness of the tyres
D. Damping of the tyres

Impact of stiffness on vehicle control

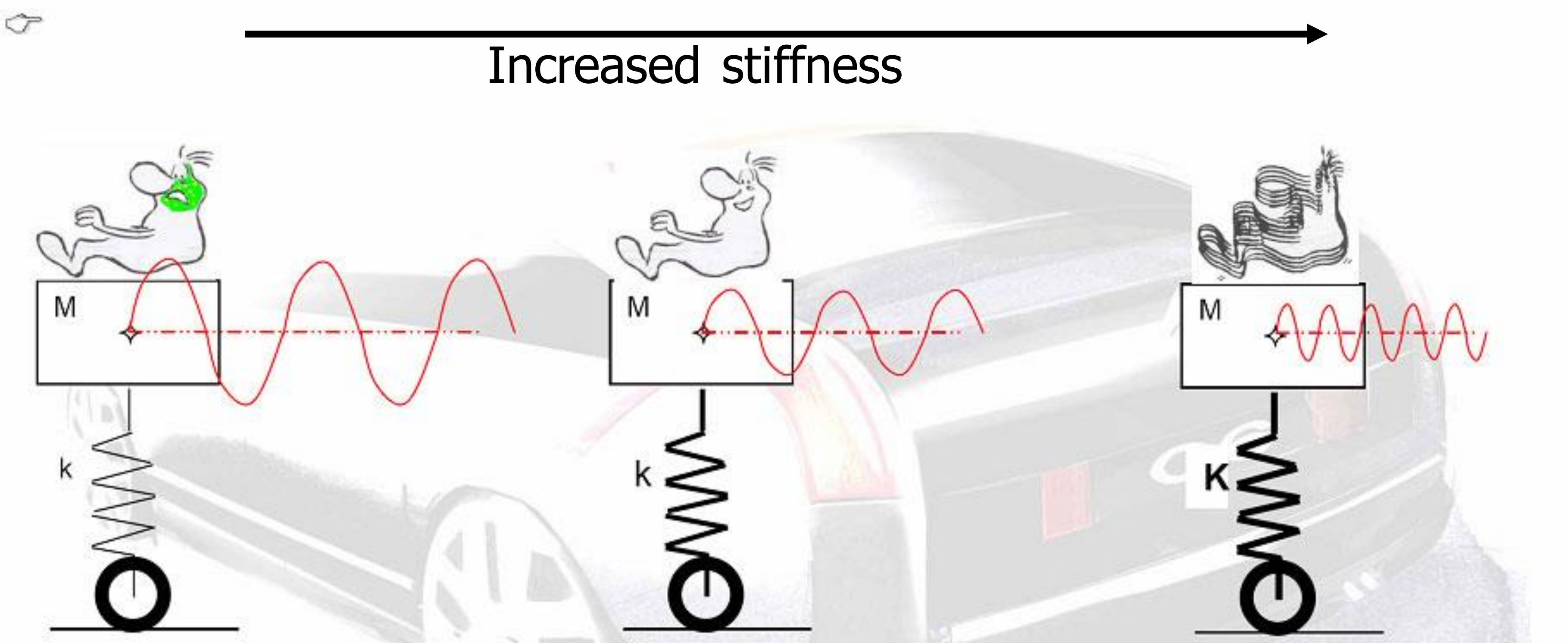
- Low stiffness
- High stiffness

The car is more sensitive for load and load variations (pump, roll and tilt)

The car is less sensitive for load and load variations (pump, roll and tilt)

A high stiffness is necessary

Impact of stiffness on comfort



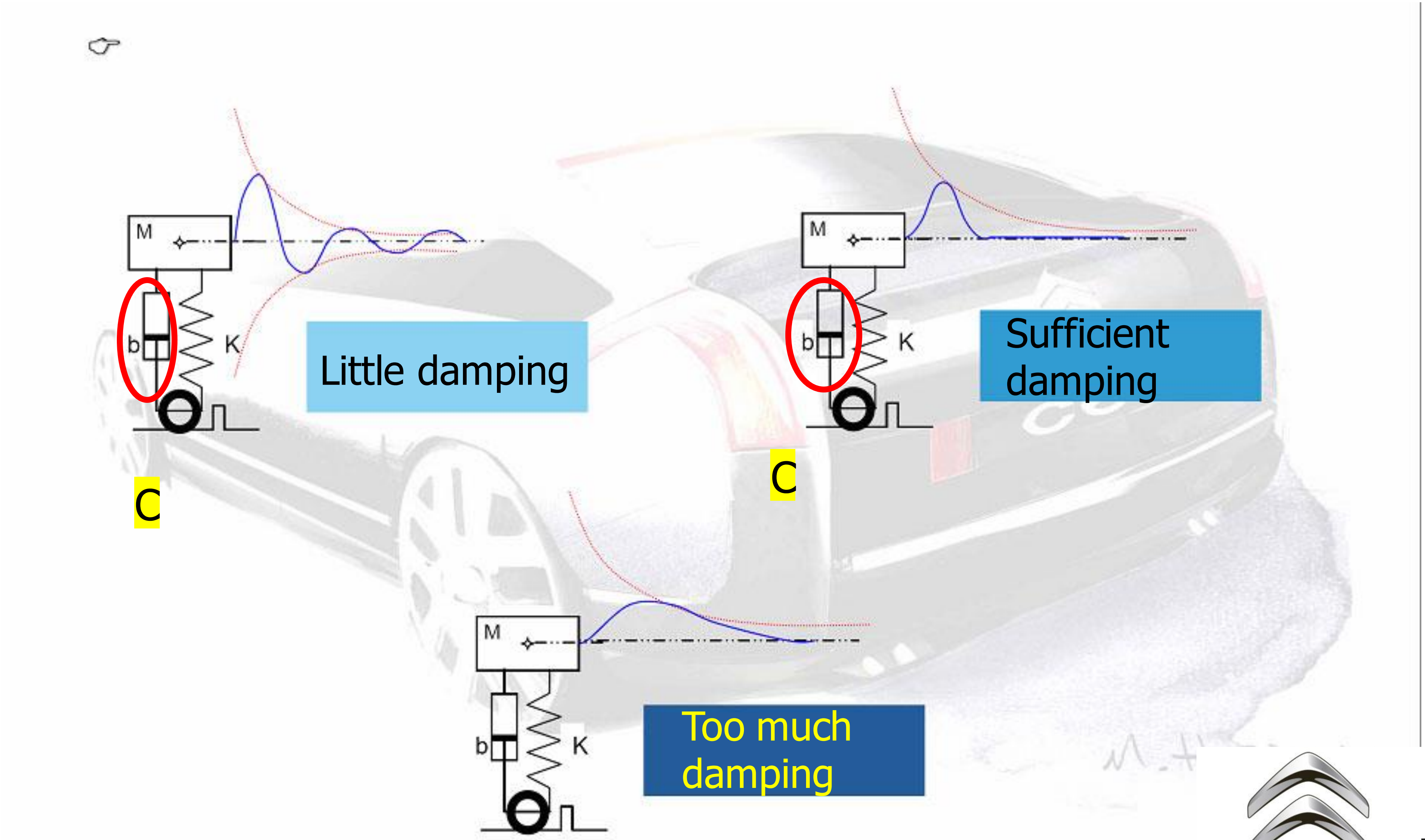
Not comfortable, boat effect. Car sickness

Just comfortable enough depending on taste

Not comfortable Racing car. Shaken passengers

The margin is small and influenced by mass, load and damping. Passive systems are never really optimal.

Impact of damping



Thank You For Your Attention!

Any Question?

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